Homework 3 — MATH 421
Practice Problems for Midterm
Wed. March 9, 2013, 7:00–8:30pm, in LGRC A301

1. Given $\alpha$ with $0 < \alpha < \pi$, find all the complex solutions $z$ to the equation
   $$z + \frac{1}{z} = 2 \cos \alpha.$$ 

2. Find all 6 solutions $z$ to the sixth-degree polynomial equation
   $$(z + 1)^7 - z^7 = 0,$$
   expressing the solutions in terms of the primitive seventh root of unity, $\omega_7$.

3. Find all the linear transformations
   $$w = az + b$$
   that map the unit disk $\{z : |z| < 1\}$ onto the disk $\{w : |w - i| < 3\}$.

4. Consider the mapping $w = (2z + 3)^2$. Determine the image in the $w$-plane of the first quadrant in the $z$-plane, namely, the open set $\{z \in \mathbb{C} : 0 < \text{Re } z < \infty, \ 0 < \text{Im } z < \infty\}$.

5.(a) Construct a single-valued branch of the function
   $$f(z) = (z - 1)^{\frac{1}{3}}$$
   that is defined on $\mathbb{C} \setminus (-\infty, 1]$, and is continuous and real-valued at all $z = x > 1$.
   (b) Are there any $z$ for which $f(z)$ takes purely imaginary values?

6. Find all solutions $z \in \mathbb{C}$ of the equation $e^{iz} = 1 + i$.

7.(a) Show that the function $f(z) = \cosh x \cos y + i \sinh x \sin y$ is an analytic function for all $z = x + iy \in \mathbb{C}$.
    (b) What is the most general function, $g(z)$, that is analytic for all $z \in \mathbb{C}$ and has the form $g(z) = \phi(x) \cos y + i\psi(x) \sin y$, for some real functions $\phi$ and $\psi$?

8. Determine $v(x, y)$ so that the complex function
   $$f(z) = \frac{x}{x^2 + y^2} + i\, v(x, y)$$
   is analytic for all $z = x + iy \neq 0$. Evaluate the complex derivative $df/dz$.
   What is this function $f(z)$?