1. Let $R$ be a domain with the field of fractions $K$. Let $F/K$ be an algebraic extension and let $S$ be the integral closure of $R$ in $F$. For any $\alpha \in F$, show that there exists $r \in R$ such that $r\alpha \in S$.

2. Suppose $n, m \geq 2$ are coprime positive integers. Show that $C[x, y]/(x^n - y^m)$ is a domain and find its normalization.

3. Let $A$ be an integrally closed domain with the field of fractions $K$. Let $F/K$ be a Galois extension with the Galois group $G$. Let $B$ be the integral closure of $A$ in $F$. Show that $G$ preserves $B$ and that $B^G = A$.
Suppose that there exist rational functions \( \theta_1(x), \ldots, \theta_n(x) \in K(x) \) such that \( \alpha_i = \theta_i(\alpha) \) for any \( i \). Suppose also that

\[ \theta_i(\theta_j(\alpha)) = \theta_j(\theta_i(\alpha)) \]

for any \( i, j \). Show that the Galois group of the splitting field of \( f(x) \) is Abelian\(^1\). (b) Give an example of the situation as in part (a) with \( K = \mathbb{Q} \) and such that the Galois group of \( f(x) \) is not cyclic. Give a specific polynomial \( f(x) \), and compute its roots and functions \( \theta_i \).

10. Let \( K = \mathbb{C}[z^{-1}, z] \) be the field of Laurent series (series in \( z \), polynomials in \( z^{-1} \)). Let \( K_m = \mathbb{C}[z^{-m}, z^m] \supset K \). (a) Show that \( K_m/K \) is Galois with a Galois group \( \mathbb{Z}/m\mathbb{Z} \). (b) Show that any Galois extension \( F/K \) with a Galois group \( \mathbb{Z}/m\mathbb{Z} \) is isomorphic to \( K_m \). (c) Show that \( K^{ab} = \bigcup_{m \geq 1} K_m \), the field of Puiseux series\(^2 \) (\( K^{ab} \) is defined in the previous worksheet).

11. Suppose \( D_4 \) acts on \( F = \mathbb{C}(x_1, \ldots, x_4) \) by permutations of variables (here we identify variables with vertices of the square). Show that \( F^{D_4} \) is generated over \( \mathbb{C} \) by 4 functions and find them.

12. Let \( F/K \) be a Galois extension with a cyclic Galois group \( G \) of order \( p \), where \( \text{char } K = p \). Let \( \sigma \) be a generator of \( G \). (a) Show that there exists \( \alpha \in F \) such that \( \sigma(\alpha) = \alpha + 1 \). (b) Show that \( F = K(\alpha) \), where \( \alpha \) is a root of \( x^p - x - a \) for some \( a \in K \).

13. Suppose that \( \text{char } K = p \) and let \( a \in K \). Show that the polynomial \( x^p - x - a \) either splits in \( K \) or is irreducible. Show that in the latter case its Galois group is cyclic of order \( p \).

14. (a) Show that

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{n-1} & \alpha_2^{n-1} & \ldots & \alpha_n^{n-1}
\end{vmatrix} = \prod_{i > j} (\alpha_i - \alpha_j).
\]

(b) For any \( k \geq 0 \), let \( p_k = \sum_{i=1}^n \alpha_i^k \). Show that

\[
\begin{vmatrix}
p_0 & p_1 & \ldots & p_{n-1} \\
p_1 & p_2 & \ldots & p_n \\
p_2 & p_3 & \ldots & p_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_n & \ldots & p_{2n-2}
\end{vmatrix} = \prod_{i > j} (\alpha_i - \alpha_j)^2.
\]

15. (continuation of the previous problem). (a) Let \( p \) be an odd prime number. Show that the discriminant of the cyclotomic polynomial \( \Phi_p(x) \) is equal to \((-1)^{\frac{p+1}{2}} p^{p-2} \). (b) Use (a) to give a different proof of the Kronecker–Weber theorem for quadratic extensions.

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\(^1\)This was proved by Abel himself.

\(^2\)Isaac Newton proved that the field of Puiseux series in fact algebraically closed.