Ars Magna or The Great Art

• A book on algebra published in 1545 by Girolamo Cardano, a physician and mathematician
• One of the most important scientific masterpieces of the early Renaissance
• Opens with introductory material on linear and quadratic equations, but then covers, for the first time, the complete procedure for solving cubic and quartic algebraic equations
• Yet, one mathematician – Tartaglia – responds to Cardano’s book with an outraged attack. What’s the story?
The Search for the Cubic

\[ ax^3 + bx^2 + cx + d = 0 \]

• The goal: to find a formula, similar to the one for the quadratic equation, that upon substitution of \(a, b, c, d\) would give the desired solution.

• Luca Pacioli in his 1494 book *Summa de arithmetica, geometria, proportioni et proportionalita*: “For the cubic and quartic [involving \(x^4\)] equations, it has not been possible until now to form general rules.”

• Solving the cubic becomes an intellectual challenge worthy of consideration by all.
Genesis of algebra

• Problems equivalent to quadratic equations were solved very early.
• Babylon (for example YBC 6967): “A number exceeds its reciprocal by \( r \). Find the number and its reciprocal.” Solved geometrically.
• Euclid, e.g. Proposition 11 from Book 2: “To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.” Solved geometrically: “Let AB be the given straight line. It is required to cut AB so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment. Describe the square ABDC on AB. Bisect AC at the point E, and join BE. Draw CA through to F, and make EF equal to BE. Describe the square FH on AF, and draw GH through to K. I say that AB has been cut at H so that the rectangle AB by BH equals the square on AH. Since the straight line AC has been bisected at E, and FA is added to it, the rectangle CF by FA together with the square on AE equals the square on EF. But EF equals EB, therefore the rectangle CF by FA together with the square on AE equals the square on EB. But the sum of the squares on BA and AE equals the square on EB, for the angle at A is right, therefore the rectangle CF by FA together with the square on AE equals the sum of the squares on BA and AE. Subtract the square on AE from each. Therefore the remaining rectangle CF by FA equals the square on AB. Now the rectangle CF by FA is FK, for AE equals FG, and the square on AB is AD, therefore FK equals AD. Subtract AK from each. Therefore FH, which remains, equals HD. And HD is the rectangle AB by BH, for AB equals BD, and FH is the square on AH, therefore the rectangle AB by BH equals the square on HA. Therefore the given straight line AB has been cut at H so that the rectangle AB by BH equals the square on HA.”

• Solution requires drawing a picture and following through complicated reasoning. Not too hard with some training but hard to conceptualize.
During the Dark Ages in Europe, mathematics flourished elsewhere: China, India, and especially in the Arab world - the nation of traders who benefitted from learning both the Indian math as well as precious Hellenistic mathematical monographs. For example, the Hindu-Arabic numerals we use today date back to that period.
Probably the most important contribution of Islamic mathematicians was introduction of “algebra”, especially in “Condensed Book on the Calculation of al-Jabr and al-Muqabala” written al-Khwarizmi (taught at the House of Wisdom in Baghdad circa 825 CE)
Al-Khwarizmi was a scholar at the House of Wisdom (circa 825 CE), a major intellectual center founded by Caliph Harun al-Rashid. Based in Baghdad from the 9th to 13th centuries, the House of Wisdom was home to many learned scholars including those of Persian or Christian background.
• The “Condensed Book on the Calculation of al-Jabr and al-Muqabala” was a manual for solving equations of all sorts on a quite abstract level.

• *al-jabr* means moving a subtracted quantity to the other side of the equation where it becomes an added quantity. In modern notation, here is an example of al-jabr: $3x+2=4-2x$ becomes $5x+2=4$.

• *al-muqabala* refers to the reduction of both sides of the equation by subtracting equal amounts from both sides: $5x+2=4$ becomes $5x=2$. 
There was no notation for unknowns: everything is explained in words. The problem might ask “What must be a square which, when increased by 10 of its own roots, amounts to 39?” (in modern notation, solve $x^2+10x=39$). The solution is explained in words equivalent to a familiar modern quadratic formula:

$$x = \frac{-10 + \sqrt{10^2 + 4 \times 39}}{2} = 3$$

One complication was absence of negative numbers. So instead of one algorithm al-Khwarizmi studies six different cases:

1. squares equal roots ($ax^2 = bx$)
2. squares equal number ($ax^2 = c$)
3. roots equal number ($bx = c$)
4. squares and roots equal number ($ax^2 + bx = c$)
5. squares and number equal roots ($ax^2 + c = bx$)
6. roots and number equal squares ($bx + c = ax^2$)
Scipione del Ferro (1465-1526)

- Son of a paper maker
- Little known of education; probably completed education at the University of Bologna (est. 1008, one of Europe’s finest universities)
- Became one of five joint holders of chair of mathematics at university (1496)
- 1515: solves the cubic equation called “the depressed cubic”, i.e. lacking its second degree term

\[ ax^3 + bx = c \]
Science in 16\textsuperscript{th} century Italy

• Unlike the 21\textsuperscript{st} century: no Internet, no journals of discoveries, no “publish or perish”
• How was a scientist’s status established?
Scientific Disputes or Duels

- Each scientist prepared questions for the other – whoever could solve the most would win.
- The scientific community was sustained through wealthy patrons, and scientists would compete in scientific disputes not only on behalf of themselves, but rather on their patrons’ behalf.
- On the results depended not only the contestant’s reputation in the city or in the University, but also tenure of appointment and increase in salary.
- At stake was not scientific creditability, but honor!
Scientific Disputes or Duels (cont.)

• So, scientific discoveries were often kept secret, to be used as a “magic” problem-solving tool in a dispute
• These were rowdy tournaments of scientific skill that could attract large crowds, including not only the students, supporters and university officials, but also spectators who came for entertainment and for betting opportunities
• Disputations took place in public squares, in churches, and in courts kept by noblemen and princes
Del Ferro therefore does not publish his discovery of the solution! On his deathbed, he divulges solution to his son-in-law, Della Nave, and to a student, Venetian Antonio del Fiore, who challenges a mathematician called Niccolò Tartaglia to a public problem-solving contest, hoping to use the cubic solution as a secret weapon.
Nicolo Tartaglia (1500-1557)

- Tartaglia was born in the northern Italian town of Brescia. His father was a mail carrier who was murdered when Tartaglia was six; his family was thrown into poverty.
- Brescia was invaded by French troops when Niccolo was twelve; the Brescian military held them off for seven days. When the Brescian forces were finally overcome, the French leader ordered to kill everyone in the town as revenge (46,000 killed).
- Tartaglia’s mother and sister hid in the local cathedral. A French soldier slashed Niccolo across the face and jaw with a saber, and left, thinking he had killed him. The wound left him with a severe scar across his face and damaged the speech apparatus.
- He was left with a bad stutter, so he was called Tartaglia, or “Stammerer.”
• Before the contest with del Fiore, Tartaglia figures out how to solve $ax^3 + bx = c$, and a day later $ax + b = x^3$, plus already knew $x^3 + ax^2 = b$

• Tartaglia easily defeats del Fiore (1535), who only knew how to solve $ax^3 + bx = c$

• Overnight, becomes the world expert on the solution of cubic equations

• From a modern perspective, these distinctions between types of equations are very minor. We treat positive and negative numbers on an equal footing. And one can always get rid of the $x^2$ term in $ax^3 + bx^2 + cx + d = 0$ by a simple substitution.
Gerolamo Cardano (1501-1576)

- Son of Milanese lawyer Fazio Cardano, who taught him arithmetic and geometry
- Studied mathematics, classics, and medicine at the universities of Pavia and Padua
- As a student, supported himself by gambling; knowledge of probability led to profits. Later writes *Liber de ludo aleae*, first book on the calculation of probabilities
- Loud voice + rude attitude → denied the doctorate of medicine, but his father’s acquaintance got him a position as a physician
- In revenge, published a book - *On the bad practices of Medicine in Common Use* - which made him one of best-known medical practitioners in Europe
• Tartaglia doesn’t publish his formula; supposedly wants to write a book on the subject
• Cardano desperately wants to get the formula from Tartaglia so that he can include it in a future book (*Ars Magna*)
The Letters Between Cardano and Tartaglia (In Brief)

- **Tartaglia** replies “Tell his Excellency that he must pardon me, that when I publish my invention it will be in my own work and not in that of others.”

- **Cardano** initially labeled Tartaglia as greedy and unwilling to help mankind, then criticized Tartaglia’s work on ballistics.

- **Tartaglia**: “But, in believing that you can demonstrate miraculously by your ridiculous opposition that I am wrong, you have only demonstrated, I will not say, that you are a great ignoramus, but that you are a person of poor judgment.”

- **Cardano**: “You should not imagine that my sharp words were caused by enmity …. I really wrote that abuse to excite you to reply.”

- **Cardano** offers to introduce Tartaglia to a wealthy patron, the Spanish viceroy and commander-in-chief in Milan, and invites Tartaglia to visit him.
What happens when Tartaglia visits?

Tartaglia gives Cardano his secret formula in cryptic verse. According to Tartaglia, the formula was so complicated that he put it into verse in order to memorize it.
What does it mean?

When the cube and things together
Are equal to some discreet number,
Find two other numbers differing in this one.
Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.
The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing
Algebraic notation was not invented yet - it appeared when mathematicians have started to abbreviate common language sentences.

| cosa (thing) | x |
| censo (square) | x^2 |
| cubo (cube) | x^3 |
| radice (root) | \(\sqrt{x}\) |

Other modern notation:
- plus → \(\text{p}\) (Luca Pacioli) → +, same for the minus
- root → R → \(\sqrt{\text{r}}\) (Euler, altered “r”) →\(\sqrt{}\) (Descartes added the vinculum)
- is equal to → = (Recorde, 1557) *To avoid the tedious repetition of these words - is equal to - I will set (as I do often in work use) a pair of parallels of one length (thus =), because no two things can be more equal.* The symbol '=' was not immediately popular. The symbol || was used by some and æ (or œ), from the Latin word aequalis meaning equal, was widely used into the 1700s.
Tartaglia’s verse in modern notation

| When the cube and things together Are equal to some discreet number | $x^3 + cx = d$ |
| Find two other numbers differing in this one. | $u - v = d$ |
| Then you will keep this as a habit That their product should always be equal Exactly to the cube of a third of the things. | $uv = \left(\frac{c}{3}\right)^3$ |
| The remainder then as a general rule Of their cube roots subtracted Will be equal to your principal thing | $x = \sqrt[3]{u} - \sqrt[3]{v}$ |
Example from Ars Magna

- $x^3 + 6x = 20$
- First solve $u - v = 20, \ uv = 2^3 = 8$
- $u = 20 + v, (20 + v)v = 8$. Quadratic equation $v^2 + 20v - 8 = 0$
  - $v = \sqrt{108} - 10, \ u = \sqrt{108} + 10$
  - $x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$

- An amusing fact: in fact $x = 2$ !!!
Cardano’s oath

- According to Tartaglia, who later publishes a “word-for-word” record of the meeting, Cardano swore that he would not publish the formula.
  - “I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only to never publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them.”

- According to Ferrari, Cardano’s student and servant who was present, Cardano did not take an oath and Tartaglia simply told it to Cardano in return for his hospitality.

- Once Cardano learned Tartaglia's solution, he succeeded not only in providing a proof for it, but he also started to work on more general cubic equations, eventually finding proofs for all cases of the cubic.
Ludovico Ferrari (1522-1565)

• First arrived at Cardano's house from Bologna as a boy of fourteen
• Cardano recognized Ferrari’s talents and took over his education.
• Ferrari was known for his temper – at age seventeen, he lost the fingers on his right hand in a knife fight.
• With Cardano's encouragement, Ferrari managed in 1540 to find a solution to the quartic equation, such as \(x^4 + 6x + 36 = 60x\).
But what about del Ferro?

- A rumor that del Ferro had left his original formula with his son-in-law reached Cardano.
- Annibale della Nave, Scipione del Ferro’s son-in-law still had the original papers of del Ferro, and in 1543 Cardano and Ferrari went to Bologna to meet with him.
- The papers proved that del Ferro had, twenty years earlier, discovered the same solution as Tartaglia had given Cardano.
- Even if Cardano had truly given an oath to Tartaglia, this was probably all he felt he needed to free himself of the obligation. The oath was formally, after all, not to reveal Tartaglia's formula, not del Ferro's.
Ars Magna

• Tartaglia did not attempt to publish his formula, most likely because he wanted to keep it to use in debates. However, by now the knowledge that such a formula existed was wide-spread.
• Cardano finally publishes the solutions to cubic and quartic equation in Ars Magna in 1545 (including the $x^3+6x=20$ example we discussed above)
• Each chapter gives a geometrical demonstration of a specific numerical cubic equation, then a verbal rule for solving that general type of equation, then one or more sample problems and equations - 13 different cubic equations with a separate chapter for each type
• In this book, Cardano explores in great detail the cubic and quartic equations and their solutions. He demonstrates for the first time that solutions can be negative, irrational, and in some cases may even involve square roots of negative numbers — quantities he refers to as "sophistic" — to be dubbed "imaginary numbers" in the seventeenth century.
• The printer Johannes Petreius of Nurnberg published the first edition of Ars magna, and it swept mathematical Europe, winning immediate acclaim.
Citation Standards

• It should be noted that citation standards in science were perhaps more lax in the 16\textsuperscript{th} century
• Tartaglia, too, was criticized for plagiarism
  – Republished, but presented William of Moerbeke’s translation of Archimedes as his own,
  – Did not credit Jordanus with the solution of the inclined plane problem
  – Proposed in the \textit{Travagliata inventione} a procedure mentioned by others for raising submerged ships.
‘Nothing is original. Steal from anywhere that resonates with inspiration or fuels your imagination. Devour old films, new films, music, books, paintings, photographs, poems, dreams, random conversations, architecture, bridges, street signs, trees, clouds, bodies of water, light and shadows. Select only things to steal from that speak directly to your soul. If you do this, your work (and theft) will be authentic. Authenticity is invaluable; originality is non-existent. And don’t bother concealing your thievery - celebrate it if you feel like it. In any case, always remember what Jean-Luc Godard said: “It’s not where you take things from - it’s where you take them to.”
Credit

However, even by modern scientific ethics, Cardano did not plagiarize, giving citations to both Tartaglia and Del Ferro.

The second paragraph in the opening chapter of *Ars Magna* reads:

“In our own days Scipione dal Ferro of Bologna has solved the case of the cube and the first power equal to a constant, a very elegant and admirable accomplishment. [...] In emulation of him, my friend Niccolo Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonia Maria Fior, and, moved by my many entreaties, gave it to me [...] Having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Ludovico Ferrari, formerly my pupil.”
Tartaglia’s Response

• Tartaglia published his own work after the Ars Magna called *Quesiti et Inventioni Diverse*, which included what he said were word-by-word accounts of the meetings, including the sacred vow.

• Tartaglia used the most offensive language to attack Cardano.

• His justification. "I truly do not know of any greater infamy than to break an oath.”
Ferrari reciprocates for Cardano

• After Tartaglia’s attack, Cardano had to defend his honor
• However, because Cardano was of a much higher socio-professional status, it would not be fitting for him to accept the challenge, so the role fell to his servant and student, Ferrari.
• Ferrari is explicit about this in his first *cartello*, or written challenge to Tartaglia: “I have decided to expose publicly your deceit or, rather, your malignant nature, not only to defend the truth, but also because this is my duty as his client, his Excellency being restrained by his status.”
Dueling Background

• The rules of the scientific disputes reflected the standards of dueling.
• Indeed, one of Ferrari’s witnesses who signed his first cartello was Mutio Iustinopolitano, or Girolamo Muzio, one of the foremost Italian experts on duels and questions of honor, and the author of *Il duello*.
  – Called by one historian “the best-known of all” treatises on dueling.
  – This shows that Cardano and Ferrari probably consulted Muzio on how to counter Tartaglia’s attack in an honorable way.
• To have Ferrari answer a challenge for Cardano was not unusual, as it was typical to pass on the task to a lower-status client.
  – Similarly, a young Kepler once supported Tycho Brahe in a dispute, and Galileo regularly used young clients to disparage attacks – this was seen as a good way to put the attackers in their places.
Battle of Insults

- Ferrari’s first *cartello*, sent on February 10, 1547, was distributed to fifty-three scholars and dignitaries across Italy.
- Ferrari: "By reading your nonsense one has the impression of reading the jokes of Piovano Arlotto [a priest who lived in the 15th century, known for his practical jokes]."
- "Among the more than one thousand errors in your book I note first that in section eight you give a result by Giordano [referring to the thirteenth-century German mathematician Jordanus Nemorarius, also known as Jordanus de Nemore] as your own, without mentioning him, and this is theft.”
Between February 10, 1547, and July 24, 1548, Tartaglia and Ferrari exchanged twelve cartelli (six challenges and six responses), all circulated to the entire intellectual high society.

Tartaglia continually attempted to drag Cardano instead of Ferrari into the dispute, wanting to dispute a man of a higher status, but Ferrari refused.

In 1548, Tartaglia was offered the position of lecturer in geometry in the home town, but the appointment was made only conditionally of him defeating Ferrari in a public contest.

Consequently, Tartaglia agrees to commit to a debate.

They agreed to sixty-two problems (thirty-one by each), which were presented in the exchanged cartelli. These were mostly in mathematics, but also in other areas such as architecture, astronomy, geography and optics.
The Dispute

• The dispute took place on August 10, 1548 in the Church in the Garden of the Frati Zoccolanti, in Milan. All of Milan’s elite, including the governor showed up.

• Ferrari brought a large entourage of his supporters, while Tartaglia may have been accompanied only by his brother. Cardano was conveniently out of town.

• The older and vastly more experienced Tartaglia was widely expected to win.

• Although we have no official record of the debate or the verdict, it seems that Tartaglia suffered a humiliating defeat.

• Tartaglia left Milan that night, leaving the contest unresolved, and the victory went to Ferrari by default.
Aftermath

- **Tartaglia**: After a year of lecturing in Brescia, he was informed that his stipend was not going to be honored, most likely as a result of his defeat in Milan, and after numerous failed lawsuits he returned to his previous teaching job in Venice. He died in poverty in Venice in 1557.

- **Ferrari's** career skyrocketed after the victory, and Ferrari even declined the offer to tutor the emperor’s son in favor of a more lucrative appointment as a tax accessor for the governor of Milan. When he returned to Bologna in 1556, he was accompanied by his sister Maddalena, a poor widow. She is widely suspected of poisoning Ferrari in 1565 – she married two weeks after his death and transferred to her husband all the money and property she had inherited from her brother.
• **Cardano** was by now a mathematical celebrity and one of the Europe’s leading scientists,– a rich and successful man. In addition to major contributions to algebra he also contributed to probability, hydrodynamics, mechanics and geology.

• Cardano’s children, however, were his great misfortune. His favorite, eldest son Giambatista married a girl Cardano called “a worthless, shameless woman” who mocked her husband for not being the father of their children. Giambatista poisoned his wife and was put into jail, tortured, and executed. Being the father of a murderer lowered Cardano’s social status, forcing him to move to Bologna, where he made many enemies.

• His other son, Aldo, became a gambler who eventually stole cash and jewelry from Cardano’s home, forcing Cardano to report Aldo to the authorities, and Aldo was banished from Bologna.
• In 1570 Cardano was put in jail on the charge of heresy – he had cast the horoscope for Christ and written a book about Nero, the tormentor of the martyrs, perhaps as an attempt to gain notoriety, since in other respects Cardano seems to have been properly religious. On his release, he was forbidden to hold a university post and barred from further publication of his work.

• Cardano went to Rome, where he received an unexpectedly warm reception. He was granted immediate membership of the College of Physicians and the Pope, who had now apparently forgiven Cardano, granted him a pension.

• He died in 1576.
So, was Cardano wrong to publish the proof?

- From Tartaglia’s perspective:
  
  – Cardano broke his oath, denying Tartaglia his recognition and fame
  – Even though Cardano gave credit, all of the references from that point on were to “Cardano’s formula”.
  – Since Cardano added so many new solutions and proofs, Tartaglia’s contribution was minimized.
  – Tartaglia’s proof was the spring board for Cardano’s own work.

- From Cardano’s perspective:
  
  – He was rescuing Tartaglia’s formula from obscurity by putting it alongside his own, groundbreaking work.
  – Tartaglia seemed to have no intent, or at least was in no rush to publish the formula.
  – Wasn’t Cardano entitled to publish his and Ferrari’s new work?
  – Del Ferro, the original discoverer, had already left the formula behind for posterity.

- What do you think? Whose intellectual property is tho formula?
Further Reading

- M. Livio, *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*
- B. Parker, *The Physics of War: From Arrows to Atoms*
- H. Hellman, *Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever*
- M. Biagioli, *Galileo, Courtier: The Practice of Science in the Culture of Absolutism*
- [http://www-history.mcs.st-andrews.ac.uk/](http://www-history.mcs.st-andrews.ac.uk/)
- *Complete Dictionary of Scientific Biography*, available through [encyclopedia.com](http://encyclopedia.com)