Ars Magna or The Great Art

- A book on algebra published in 1545 by Girolamo Cardano, a physician and mathematician
- One of the most important scientific masterpieces of the early Renaissance
- Opens with introductory material on linear and quadratic equations, but then covers, for the first time, the complete procedure for solving cubic and quartic algebraic equations
- Yet, one mathematician – Tartaglia – responds to Cardano’s book with an outraged attack. What’s the story?
The Search for the Cubic

\[ ax^3 + bx^2 + cx + d = 0 \]

• The goal: to find a formula, similar to the one for the quadratic equation, that upon substitution of \( a, b, c, d \) would give the desired solution

• Luca Pacioli in his 1494 book *Summa de arithmetica, geometria, proportioni et proportionalita*: “For the cubic and quartic [involving \( x^4 \)] equations, it has not been possible until now to form general rules.”

• Solving the cubic becomes an intellectual challenge worthy of consideration by all
Genesis of algebra

- Problems equivalent to quadratic equations were solved very early.
- Babylon (for example YBC 6967): “A number exceeds its reciprocal by \( r \). Find the number and its reciprocal.” Solved geometrically.
- Euclid, e.g. Proposition 11 from Book 2: “To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.” Solved geometrically: “Let \( AB \) be the given straight line. It is required to cut \( AB \) so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment. Describe the square \( ABDC \) on \( AB \). Bisect \( AC \) at the point \( E \), and join \( BE \). Draw \( CA \) through to \( F \), and make \( EF \) equal to \( BE \). Describe the square \( FH \) on \( AF \), and draw \( GH \) through to \( K \). I say that \( AB \) has been cut at \( H \) so that the rectangle \( AB \) by \( BH \) equals the square on \( AH \). Since the straight line \( AC \) has been bisected at \( E \), and \( FA \) is added to it, the rectangle \( CF \) by \( FA \) together with the square on \( AE \) equals the square on \( EF \). But \( EF \) equals \( EB \), therefore the rectangle \( CF \) by \( FA \) together with the square on \( AE \) equals the square on \( EB \). But the sum of the squares on \( BA \) and \( AE \) equals the square on \( EB \), for the angle at \( A \) is right, therefore the rectangle \( CF \) by \( FA \) together with the square on \( AE \) equals the sum of the squares on \( BA \) and \( AE \). Subtract the square on \( AE \) from each. Therefore the remaining rectangle \( CF \) by \( FA \) equals the square on \( AB \). Now the rectangle \( CF \) by \( FA \) is \( FK \), for \( AE \) equals \( FG \), and the square on \( AB \) is \( AD \), therefore \( FK \) equals \( AD \). Subtract \( AK \) from each. Therefore \( FH \), which remains, equals \( HD \). And \( HD \) is the rectangle \( AB \) by \( BH \), for \( AB \) equals \( BD \), and \( FH \) is the square on \( AH \), therefore the rectangle \( AB \) by \( BH \) equals the square on \( HA \). Therefore the given straight line \( AB \) has been cut at \( H \) so that the rectangle \( AB \) by \( BH \) equals the square on \( HA \).”
- Solution requires drawing a picture and following through complicated reasoning. Not too hard with some training but hard to conceptualize.
During the Dark Ages in Europe, mathematics flourished elsewhere: China, India, and especially in the Arab world - the nation of traders who benefitted from learning both the Indian math as well as precious Hellenistic mathematical monographs. For example, the Hindu-Arabic numerals we use today date back to that period.
Probably the most important contribution of Islamic mathematicians was introduction of “algebra”, especially in “Condensed Book on the Calculation of al-Jabr and al-Muqabala” written al-Khwarizmi (taught at the House of Wisdom in Baghdad circa 825 CE)
House of Wisdom

Al-Khwarizmi was a scholar at the House of Wisdom (circa 825 CE), a major intellectual center founded by Caliph Harun al-Rashid. Based in Baghdad from the 9th to 13th centuries, the House of Wisdom was home to many learned scholars including those of Persian or Christian background.
• The “Condensed Book on the Calculation of al-Jabr and al-Muqabala” was a manual for solving equations of all sorts on a quite abstract level.

• *al-jabr* means moving a subtracted quantity to the other side of the equation where it becomes an added quantity. In modern notation, here is an example of al-jabr: \( 3x+2=4-2x \) becomes \( 5x+2=4 \).

• *al-muqabala* refers to the reduction of both sides of the equation by subtracting equal amounts from both sides: \( 5x+2=4 \) becomes \( 5x=2 \).
There was no notation for unknowns: everything is explained in words. The problem might ask “What must be a square which, when increased by 10 of its own roots, amounts to 39?” (in modern notation, solve $x^2+10x=39$). The solution is explained in words equivalent to a familiar modern quadratic formula

$$x = \frac{-10 + \sqrt{10^2 + 4 \times 39}}{2} = 3$$

One complication was absence of negative numbers. So instead of one algorithm al-Khwarizmi studies six different cases

(1) squares equal roots ($ax^2 = bx$)
(2) squares equal number ($ax^2 = c$)
(3) roots equal number ($bx = c$)
(4) squares and roots equal number ($ax^2 + bx = c$)
(5) squares and number equal roots ($ax^2 + c = bx$)
(6) roots and number equal squares ($bx + c = ax^2$)
Scipione del Ferro (1465-1526)

• Son of a paper maker
• Little known of education; probably completed education at the University of Bologna (est. 1008, one of Europe’s finest universities)
• Became one of five joint holders of chair of mathematics at university (1496)
• 1515: solves the cubic equation called “the depressed cubic”, i.e. lacking its second degree term

\[ ax^3 + bx = c \]
Science in 16\textsuperscript{th} century Italy

• Unlike the 21\textsuperscript{st} century: no Internet, no journals of discoveries, no “publish or perish”
• How was a scientist’s status established?
Scientific Disputes or Duels

• Each scientist prepared questions for the other – whoever could solve the most would win.
• The scientific community was sustained through wealthy patrons, and scientists would compete in scientific disputes not only on behalf of themselves, but rather on their patrons’ behalf
• On the results depended not only the contestant’s reputation in the city or in the University, but also tenure of appointment and increase in salary
• At stake was not scientific creditability, but honor!
So, scientific discoveries were often kept secret, to be used as a “magic” problem-solving tool in a dispute.

These were rowdy tournaments of scientific skill that could attract large crowds, including not only the students, supporters and university officials, but also spectators who came for entertainment and for betting opportunities.

Disputations took place in public squares, in churches, and in courts kept by noblemen and princes.
Del Ferro therefore does not publish his discovery of the solution! On his deathbed, he divulges solution to his son-in-law, Della Nave, and to a student, Venetian Antonio del Fiore, who challenges a mathematician called Niccolo Tartaglia to a public problem-solving contest, hoping to use the cubic solution as a secret weapon.
Nicolo Tartaglia (1500-1557)

- Tartaglia was born in the northern Italian town of Brescia. His father was a mail carrier who was murdered when Tartaglia was six; his family was thrown into poverty.
- Brescia was invaded by French troops when Niccolo was twelve; the Brescian military held them off for seven days. When the Brescian forces were finally overcome, the French leader ordered to kill everyone in the town as revenge (46,000 killed).
- Tartaglia’s mother and sister hid in the local cathedral. A French soldier slashed Niccolo across the face and jaw with a saber, and left, thinking he had killed him. The wound left him with a severe scar across his face and damaged the speech apparatus.
- He was left with a bad stutter, so he was called Tartaglia, or “Stammerer.”
• Before the contest with del Fiore, Tartaglia figures out how to solve $ax^3 + bx = c$, and a day later $ax + b = x^3$, plus already knew $x^3 + ax^2 = b$

• Tartaglia easily defeats del Fiore (1535), who only knew how to solve $ax^3 + bx = c$

• Overnight, becomes the world expert on the solution of cubic equations

• From a modern perspective, these distinctions between types of equations are very minor. We treat positive and negative numbers on an equal footing. And one can always get rid of the $x^2$ term in $ax^3 + bx^2 + cx + d = 0$ by a simple substitution.
Further Reading

• M. Livio, *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*
• P. Pesic, *Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability*
• B. Parker, *The Physics of War: From Arrows to Atoms*
• H. Hellman, *Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever*
• M. Biagioli, *Galileo, Courtier: The Practice of Science in the Culture of Absolutism*
• [http://www-history.mcs.st-andrews.ac.uk/](http://www-history.mcs.st-andrews.ac.uk/)
• *Complete Dictionary of Scientific Biography*, available through [encyclopedia.com](http://encyclopedia.com)