The Probability of Winning at Craps Is $244/495 = 0.492929$

Here are the rules of craps.

If you roll a total of 7 or 11 on the first roll, you win.

If you roll a total of 2, 3, or 12 on the first roll, you lose.

If you roll a total of 4, 5, 6, 8, 9, or 10 on your first roll, this number becomes your point. You continue to roll the dice. If you get your point total before a total of 7 appears, you win. If you roll a total of 7 before your point total appears, you lose.

Note: This is a 1:1 bet. That is, when you win, you win a dollar for each dollar that you bet.

The possible totals obtained from rolling two dice are shown at the right. Note that there are 36 cells containing the totals, and each cell has a probability of $1/36$ of being the result of a craps roll.

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**Notation.** $(\alpha_j, \beta_j) = (\text{die 1}, \text{die 2})$ on $j^{\text{th}}$ roll, $1 \leq \alpha_j \leq 6, 1 \leq \beta_j \leq 6$.

For $2 \leq k \leq 12$ define $p_1(k)$ to be the probability of obtaining $k$ in a roll of two dice. Thus

$$p_1(k) = P(\{\alpha_1, \beta_1\}: \alpha_1 + \beta_1 = k).$$

The table at the top of the page shows that

- $p_1(k) = 1/36$ for $k = 2$ or 12,
- $p_1(k) = 2/36$ for $k = 3$ or 11,
- $p_1(k) = 3/36$ for $k = 4$ or 10,
- $p_1(k) = 4/36$ for $k = 5$ or 9,
- $p_1(k) = 5/36$ for $k = 6$ or 8,
- $p_1(k) = 6/36$ for $k = 7$.

On the next page we calculate the probability of winning.
Sample space \( S^{(\infty)} = \{ (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots : 1 \leq \alpha_i, \beta_i \leq 6, i \in \mathbb{N} \} \). Denote the outcomes in \( S^{(\infty)} \) by \( x = (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots \).

Define \( W = \{ x \in S^{(\infty)} : x \text{ wins} \}, \)
\[
W_7 = \{ x \in S^{(\infty)} : \alpha_1 + \beta_1 = 7 \} \quad \text{[7 on first roll]}
\]
\[
W_{11} = \{ x \in S^{(\infty)} : \alpha_1 + \beta_1 = 11 \} \quad \text{[11 on first roll]}
\]

For \( k = 4, 5, 6, 8, 9, 10 \), define \( W_k = \{ x \in S^{(\infty)} : \alpha_1 + \beta_1 = k \} \), \( \text{[} k \text{ on first roll} \] \)
\[
W_{k,k}^{(2)} = \{ x \in S^{(\infty)} : \alpha_1 + \beta_1 = k, \alpha_2 + \beta_2 = k \} \quad \text{[point \( \neq k \) and obtain \( k \) on second roll]}
\]

and for \( n \geq 3 \) \( \text{[point \( \neq k \), obtain neither \( k \) nor 7 on rolls 2, 3, \ldots, } n-1, \text{ obtain } k \text{ on roll } n \] \)
\[
W_{k,k}^{(n)} = \{ x \in S^{(\infty)} : \alpha_1 + \beta_1 = k, \alpha_2 + \beta_2 \neq k \text{ or 7}, \ldots, \alpha_{n-1} + \beta_{n-1} \neq k \text{ or 7}, \alpha_n + \beta_n = k \}
\]

Then
\[
W = W_7 \cup W_{11} \cup \bigcup_{4 \leq k \leq 6} W_{k,k}^{(2)} \cup \bigcup_{8 \leq k \leq 10} W_{k,k}^{(n)}
\]

Since these events are mutually exclusive,
\[
P(W) = P(W_7) + P(W_{11}) + \sum_{4 \leq k \leq 6} \sum_{n=2}^{\infty} P(W_{k,k}^{(n)})
\]  
\[+ \sum_{8 \leq k \leq 10} \sum_{n=2}^{\infty} P(W_{k,k}^{(n)})
\]
\[= P(W_7) + P(W_{11}) + 2 \sum_{4 \leq k \leq 6} \sum_{n=2}^{\infty} P(W_{k,k}^{(n)})
\]
The last step holds since \( P(W_{4,4}^{(n)}) = P(W_{10,10}^{(n)}) \), \( P(W_{5,5}^{(n)}) = P(W_{9,9}^{(n)}) \), and \( P(W_{6,6}^{(n)}) = P(W_{8,8}^{(n)}) \).
The sets $W_1, W_{II},$ and $W_{k,k}$ depend on infinitely many rolls of the dice. How should we define the probabilities of these sets? For consistency, we must have $P(\{x\}) = 0$ for all $x \in S^{(\infty)}$.

For any $X = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots) \in S^{(\infty)}$ and $n \in \mathbb{N}$, define

$$\pi_n X = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n).$$

Define for $E \subset S^{(\infty)}$

$$\pi_n S^{(\infty)} = \{\pi_n X : X \in S^{(\infty)}\}$$

$$\pi_n E = \{y \in \pi_n S^{(\infty)} : y = \pi_n X \text{ for } x \in E\}$$

Examples:

1) $\pi_1 W_1^1 = \{(\alpha_1, \beta_1) : \alpha_1 + \beta_1 = 73\}$
2) $\pi_1 W_{II} = \{(\alpha_1, \beta_1) : \alpha_1 + \beta_1 = 11\}$
3) $\pi_k W_k = \{(\alpha_1, \beta_1) : \alpha_1 + \beta_1 = k\}$ for $4 \leq k \leq 6$.
4) $\pi_2 W_{k,k}^{(2)} = \{(\alpha_1, \beta_1, \alpha_2, \beta_2) : \alpha_1 + \beta_1 = k, \alpha_2 + \beta_2 = k\}$
5) $\pi_3 W_{k,k}^{(3)} = \{(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) : \alpha_1 + \beta_1 = k, \alpha_2 + \beta_2 \neq k \text{ or } 7, \alpha_3 + \beta_3 = k\}$

We say that $E \subset S^{(\infty)}$ depends on the first $n$ rolls if one can decide that $x \in E$ by examining $\pi_n x$.

Examples:

1) $W_1, W_{II},$ and $W_{k,k}$ depend on the first roll,
2) $W_{k,k}^{(2)}$ depends on the first 2 rolls,
3) $W_{k,k}^{(3)}$ for $n \geq 3$ depends on the first $n$ rolls.
For any \( y \in \pi_n S(\omega) \) define \( P_n(\{y\}) = \frac{1}{36^n} \).

For \( E \subset S(\omega) \) depending on the first \( n \) rolls, define

\[
P(E) = P_n(\pi_n E) = \sum_{y \in \pi_n E} P_n(\{y\}) = \frac{|\pi_n E|}{36^n}.
\]

**Examples**

\[
P(W_7) = P(\pi_1 W_7) = \frac{|\pi_1 W_7|}{36} = \frac{6}{36} = \frac{1}{6},
\]

\[
P(W_{11}) = P(\pi_1 W_{11}) = \frac{|\pi_1 W_{11}|}{36} = \frac{2}{36},
\]

\[
P(W_k) = P(\pi_1 W_k) = \frac{|\pi_1 W_k|}{36} = \frac{a_k}{36}, \quad a_4 = 3, \quad a_5 = 4, \quad a_6 = 5
\]

\[
P(W_{k,2}) = P(\pi_2 W_{k,2})
\]

\[
= P(\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}) : \alpha_1 + \beta_1 = k, \quad \alpha_2 + \beta_2 = k
\]

\[
= P(\{k\}) \cdot P(\{k\}) = \frac{a_k}{36} \cdot \frac{a_k}{36} \quad [\text{by independence}]
\]

\[
P(W_{k,k}) = P(\pi_n W_{k,k})
\]

\[
= P(\{\alpha_1, \beta_1\}, \ldots, \{\alpha_n, \beta_n\}) : \alpha_1 + \beta_1 = k, \alpha_j + \beta_j \neq k \text{ for } 2 \leq j \leq n - 1, \alpha_n + \beta_n = k
\]

\[
= P(\{k\}) \left[1 - P(\{k\}) - P(\{\bar{k}\})\right]^{n-2} P(\{k\}) \quad [\text{by independence}]
\]

\[
= \frac{a_k}{36} \cdot \frac{a_k}{36} \cdot \left(1 - \frac{6 + a_k}{36}\right)^{n-2} \text{ for } n \geq 2
\]

Make sure that you understand how independence is used to calculate \( P_n(\pi_n W_{k,k}) \).
\[ P(W) = P(W_7) + P(W_8) + 2 \sum_{k=4}^{\infty} \sum_{n=2}^{\infty} P(W_{n,k}) \]

\[ = \frac{6}{36} + \frac{2}{36} + 2 \sum_{k=4}^{\infty} \frac{a_k^2}{36^2} \sum_{j=0}^{\infty} \left(1 - \frac{6+q_k}{36}\right)^j \]

\[ = \frac{6}{36} + \frac{2}{36} + 2 \sum_{k=4}^{\infty} \frac{a_k^2}{36^2} \frac{1}{1-(1-\frac{6+q_k}{36})} \]

\[ = \frac{6}{36} + \frac{2}{36} + 2 \sum_{k=4}^{\infty} \frac{a_k^2}{36(6+q_k)} \]

\[ = \frac{6}{36} + \frac{2}{36} + \left[\frac{3^2}{36 \cdot 9} + \frac{4^2}{36 \cdot 10} + \frac{5^2}{36 \cdot 11}\right] \]

\[ = \frac{2}{9} + \frac{1}{18} + \frac{4}{95} + \frac{25}{198} = \frac{2 \cdot 25 \cdot 9 \cdot 11 + 5 \cdot 11 + 8 \cdot 11 + 25 \cdot 5}{2 \cdot 5 \cdot 9 \cdot 11} \]

\[ = \frac{220 + 55 + 88 + 125}{990} = \frac{488}{990} = \frac{244}{495} = 0.492929 \]