Poisson Limit Theorems and Poisson Approximation

In Theorems 1 and 2 we fix \( \lambda > 0 \) and obtain the Poisson distribution for \( \lambda \).

As a limit of the binomial distribution of \( S_n, p_n \), where \( p_n = \lambda/n \),
in Theorem 1 and \( n p_n \rightarrow \lambda \) in Theorem 2. In Theorems 3 and 4 we fix \( n \) and \( p \) and obtain an approximation to the Poisson distribution for \( \lambda = n p \) by the binomial distribution of \( S_n, p \).

**Theorem 1.** Given \( \lambda > 0 \) let \( S_n, p_n \) be a sequence of binomial rv's with parameters \( n \) and \( \lambda/n \). Let \( N \) be a Poisson rv with parameter \( \lambda \).

Then for any \( i \in \mathbb{N} \cup \{0\} \)

\[
\lim_{n \to \infty} P(S_n = \lambda/n i) = e^{-\lambda} \frac{\lambda^i}{i!} = P(N = i). \quad \text{[Proved in class]}
\]

**Theorem 2.** Same notation as in Theorem 1 with \( \lambda/n \) replaced by \( p_n \),

where \( n p_n \rightarrow \lambda \). Then for any \( i \in \mathbb{N} \cup \{0\} \)

\[
\lim_{n \to \infty} P(S_n = \lambda/n i) = e^{-\lambda} \frac{\lambda^i}{i!} = P(N = i). \quad \text{[If \( p_n = \lambda/n \), then get Thm 1]}
\]

**Theorem 3.** Given \( n \in \mathbb{N} \) and \( 0 < p \leq 1 \), let \( S_n, p \) be a binomial rv with parameter \( n \) and \( p \). Let \( N_{np} \) be a Poisson rv with parameter \( \lambda = np \). Then for any \( i \in \mathbb{N} \cup \{0\} \)

\[
|P(S_{np} = i) - P(N_{np} = i)| \leq np^2. \quad \text{[Proved in 58.6 of text]}
\]

If \( p = \lambda/n \), then Thm 3 yields Thm 1 because \( np^2 = \frac{\lambda^2}{n} \rightarrow 0 \).

In general, this approximation is useful if \( np^2 \) is small;
similarly in Theorem 4.

**Theorem 4.** Same notation as in Theorem 3. Then for any

subset \( A \subset \mathbb{N} \cup \{0\} \)

\[
|P(S_{np} \in A) - P(N_{np} \in A)| = \left| \sum_{i \in A} P(S_{np} = i) - \sum_{i \in A} P(N_{np} = i) \right| \leq np^2.
\]

[Proved in 58.6 of text]