Notes 4: Parametrization

A parametrization of a curve or a surface is a map from $\mathbb{R}, \mathbb{R}^2$ to the curve or surface that covers almost all of the surface.

Example 1. We can parametrize the circle $x^2 + y^2 = 9$ using trig functions:

$$t \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \cos(t) \\ 3 \sin(t) \end{pmatrix}.$$  

Example 2. Here is a parametrization of the circle $x^2 + y^2 = 1$ by rational functions:

$$t \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-t^2 \\ \frac{1+t^2}{1+t^2} \end{pmatrix}.$$  

Note in this example that the point $(x = -1, y = 0)$ is not in the image of the parametrization. This is an example where the parametrization fails to cover the whole curve. Later we will make clear what we mean by 'almost all'. We say this is a parametrization because the functions are rational functions. A polynomial parametrization would be one where all the functions are polynomials.

Example 3. We start with the twisted cubic, the curve in $\mathbb{R}^3$ defined as the locus of zeros of the two polynomials $y - x^2$ and $z - x^3$. We wish to parametrize the surface consisting of the union of all the tangent lines to the twisted cubic. We start by parametrizing the twisted cubic. Note that given $x$ we have $y = x^2, z = x^3$. This leads to the parametrization

$$t \mapsto \begin{cases} x = t \\ y = t^2 \\ z = t^3. \end{cases}$$

We construct the tangent line to the twisted curve at the point $(x = t, y = t^2, z = t^3)$. The tangent vector at this point is $(1, 2t, 3t^2)$. We can parametrize the tangent vector by

$$t \mapsto \begin{cases} x = t + s \cdot 1 \\ y = t^2 + s \cdot 2t \\ z = t^3 + s \cdot 3t^2. \end{cases}$$

As $s, t$ vary we cover the entire surface with this parametrization.

Question: Is this an affine variety?

The answer is yes. Using the first two equations we can solve for $s, t$ and express $z$ as a rational function of $x$ and $y$ with no $s$ or $t$. Clearing fractions we get the desired polynomial equation

$$-4x^3z + 3x^2y^2 - 4y^3 + 6xyz - z^2 = 0.$$  

Warning: When we clear fractions we can introduce extraneous solutions. An example is $\frac{x}{y} = z$. Clearing fractions gives $x = yz$ with a whole line of extra solutions $y = 0, z = \text{anything}, x = 0$. 

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Example 4. A basic calculation in linear algebra can be viewed as finding a parametrization of a subspace of $\mathbb{R}^n$. Given a matrix $M$ of size $n \times m$, so $M$ defines a map

$$M : \mathbb{R}^m \to \mathbb{R}^n,$$

we have an algorithm for finding a basis of the kernel of $M$. This gives a polynomial (actually a linear one) parametrization of the subspace $\ker(M)$ of $\mathbb{R}^m$.

We do an example. Let

$$M = \begin{pmatrix} 1 & 2 & 1 & -1 \\ -1 & -3 & 3 & 4 \end{pmatrix}.$$ 

This means $M$ defines a map from $\mathbb{R}^4$ to $\mathbb{R}^2$. The kernel of $M$ is a subspace of $\mathbb{R}^4$. We want a parametrization of the kernel of $M$, that is, the variety defined by

$$x + 2y + z - w = 0,$$
$$-x - 3y + 2z + 4w = 0.$$

We perform elementary row operations on $M$ to obtain

$$\begin{pmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 3 \end{pmatrix}.$$ 

This is equivalent to the equations

$$x + 7z + 5w = 0,$$
$$y + 3z + 3w = 0.$$

Solving we get

$$x = -7z - 5w,$$
$$y = -3z - 3w,$$
$$z = \text{anything},$$
$$w = \text{anything}.$$

We can rewrite this as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -7 \\ -5 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$ 

Thus we have parametrized the kernel of $M$. Note that the two vectors

$$\begin{pmatrix} 7 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

are a basis of the kernel of $M$. 

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