Chasles’ Theorem: Let $C_1, C_2, C_3$ be cubic curves in $\mathbb{P}^2$. Assume that $C_1 \cap C_2$ is a set of 9 distinct points $\Gamma = \{P_1, \ldots, P_9\}$. If $C_3$ passes through 8 of these, then $C_3$ passes through the ninth point.

Terminology. If $V$ is a subspace of a finite dimensional vector space $W$, then the codimension of $V$ in $W$ is $\dim(W) - \dim(V)$. Let $S_d$ denote the space of homogeneous polynomials on $\mathbb{P}^2$ of degree $d$. If $\Gamma$ in a set of $m$ points in $\mathbb{P}^2$, we say it imposes $l$ conditions on $S_d$, if the subspace of $S_d$ consisting of forms vanishing on $\Gamma$ has codimension $l$ in $S_d$.

We can restate Chasles’ Theorem as:
Let $\Gamma = \{P_1, \ldots, P_9\}$ be the intersection of two cubics and let $\Gamma' = \{P_1, \ldots, P_8\}$. Then $\Gamma$ and $\Gamma'$ impose the same number of conditions on $S_3$, the space of cubic forms on $\mathbb{P}^2$.

We actually prove a stronger statement:
The sets $\Gamma$ and $\Gamma'$ both impose 8 conditions on $S_3$.

Remark: The space of homogeneous forms in $X, Y, Z$ of degree 3 has dimension 10.

Exercise: Give a basis for this space.
Let $\Gamma = \{P_1, \ldots, P_9\} = C_1 \cap C_2$ as above. I claim that $\Gamma$ fails to impose 9 conditions on $S_3$. Let $C_1, C_2$ be the zero loci of homogeneous forms $F_1$ and $F_2$ of degree 3. Then $\alpha F_1 + \beta F_2, \alpha, \beta \in \mathbb{C}$ forms a two dimensional subspace of $S_3$ of forms that vanish on $\Gamma$. Hence the codimension of the space of forms vanishing on $\Gamma$ is $\leq 8$.

Terminology: A set of $m$ points in $\mathbb{P}^2$ fails to impose independent conditions on $S_d$ if the space of forms in $S_d$ vanishing on the set of $m$ points has codimension strictly less than $m$.

We investigate what it means for a set of points to fail to impose independent conditions on homogeneous forms of degree 1, 2, 3.

Lemma 1: Let $\Omega$ be a set of $n \leq 4$ points. Then $\Omega$ fails to impose linearly independent conditions on forms of degree 1 if and only if 3 of the points are collinear or $n = 4$.

Proof: If $n = 1$, then the condition that a form $aX + bY + cZ$ vanish at a given point is non-trivial, hence the space of triples $(a, b, c)$ so that the corresponding form vanishes at the given point in codimension 1.

Note that the space of homogeneous forms in $X, Y, Z$ of degree 1 has dimension 3. If $n = 2$, then there is a unique line passing through two points. Hence the space
of forms vanishing on $\Omega$ has codimension 2.

If $n = 3$, then one of two things can occur:
a: The points are collinear. In this case there is a 1 dimensional family of forms of degree 1 vanishing on $\Omega$. This has codimension 2 and hence $\Omega$ fails to impose linearly independent conditions.

b: The points are not collinear. Then there are no linear forms vanishing on $\Omega$ and the codimension is 3. Thus $\Omega$ does impose linearly independent conditions in this case.

If $n = 4$, then $\Omega$ fails to impose independent conditions because the space of forms of degree 1 is dimension 3, so the codimension of the space of forms vanishing on $\Omega$ has codimension $\leq 3 < 4$.

Criterion: Assume we are given a set of $m$ points and let a subset of $m - 1$ points imposes independent conditions on $S_d$. If there is a form of degree $d$ that vanishes on the subset of $m - 1$ points and does not vanish on the last point, then the set of $m$ points imposes linearly independent conditions.

Exercise: Explain why.

Lemma 2: Let $\Omega = \{P_1, \ldots, P_n\}$ be a collection of $n$ points in $\mathbb{P}^2$ with $n \leq 6$. The set $\Omega$ fails to impose independent conditions on $S_2$, the space of forms of degree 2, if and only if either 4 points are collinear or $n = 6$ and $\Omega$ is contained in a conic.

We first prove the backward implication $\Leftarrow$: Assume that 4 points among the $n$ points in $\Omega$ lie on the line defined by the vanishing of the linear form $H$. The set of conics vanishing on these 4 points is the set of conics of the form $H \cdot G$ with $G$ linear. This is a space of dimension 3 (Exercise: Why?) and hence these 4 points impose $6 - 3 = 3$ conditions on conics. The remaining $n - 4$ points impose at most $n - 4$ conditions. Thus $\Omega$ imposes $\leq 3 + n - 4 = n - 1$ conditions on conics.

Assume that $n = 6$ and $\Omega$ lies on a conic $F = 0$. Thus there is at least 1 dimensional space of conics vanishing on $\Omega$. Hence the codimension of such forms is $\leq 5$. Thus $\Omega$ fails to impose independent conditions.

We prove the forward implication $\Rightarrow$:
If $n = 1$, then $\Omega = \{P\}$. The point $P$ does impose 1 condition on $S_2$.

Assume $n = 2$: Given two points $\{P_1, P_2\} = \Omega$ there is a pair of lines (and hence an element of $S_2$) that passes through $P_1$ and not $P_2$. Now apply our criterion to conclude that $\Omega$ imposes independent conditions.
Exercise: Assume that $P_1 = (1, -2, 3)$ and $P_2 = (2, 0, 1)$. Give the equation of a line that passes through $P_1$, but not through $P_2$.

$n = 3$: There are two cases. If $\{P_1, P_2, P_3\}$ are collinear we can find a conic through two of the points and not the third. If they are not collinear, then we can find a pair of lines through two but not the third:

$n = 4$: We need to show that if 4 points $\{P_1, P_2, P_3, P_4\} = \Omega$, then they impose independent conditions if they are not collinear. To do this we use our criterion. In particular, we show that there is a conic through three of the points and not the fourth. See the pictures below.

Now the criterion applies.

$n = 5$: Assume no 4 points of $\Omega = \{P_1, P_2, \ldots, P_5\}$ are collinear. We show we can find a conic through 4 and not the 5th. If no 3 of the points are collinear:

If 3 points are collinear:

$n = 6$: Need to show that if $\{P_1, \ldots, P_6\}$ does not lie on a conic and no 4 of these points lie on a line, then $\{P_1 \ldots P_6\}$ imposes independent conditions. Since the family of homogeneous polynomials of degree 2 in variables $X, Y,$ and $Z$ is 6 dimensional, this means we have to show that no non-trivial such polynomial vanishes at all 6 points.
If no 3 of the points \( \{P_1 \ldots P_5\} \) are collinear, then there is a unique polynomial of degree two vanishing on \( P_1 \ldots P_5 \). By hypothesis \( P_6 \) is not on this conic, hence there are no non-trivial degree 2 forms vanishing on \( P_1, P_2, \ldots P_6 \).

Assume \( P_1, P_2, P_3 \) are collinear; say on the line \( L \). By hypothesis none of \( P_4, P_5, P_6 \) are on \( L \). The points \( P_4, P_5, P_6 \) cannot lie on a line \( M \), for if they did, \( P_1, \ldots P_6 \) would lie on the conic formed as the union of \( L \) and \( M \) thus \( \{P_1, \ldots P_6\} \) would fail to satisfy our hypothesis. Any conic passing through \( P_1, \ldots P_6 \) contains the line \( L \) (why?) and hence is the union of two lines. But this is impossible. Hence no conic passes through \( \{P_1, \ldots P_6\} \).

Lemma 3: Let \( \{P_1, \ldots P_n\} = \Omega \) be a collection of \( n \) points in \( \mathbb{P}^2 \) with \( n \leq 8 \). The points \( \Omega \) fail to impose independent conditions on curves of degree 3 if and only if 5 points in \( \Omega \) lie on a line or \( n = 8 \) and \( \{P_1, \ldots P_6\} \) lies on a conic.

Proof: \( \Leftarrow: \) The dimension of the space of forms of degree 2 is 6 and the dimension of the space of forms of degree 3 is 10.

If a form \( F \) of degree 3 vanishes at more than 3 points on a line \( L \), then \( F = G \cdot H \) where \( L \) is the zero locus of the linear form \( G \) and \( H \) is a form of degree 2. This follows from Bezout’s theorem.

If 5 points of \( \Omega \) lie on a line defined by a linear form \( G \), then any cubic form vanishing on \( \Omega \) can be written \( F = GH \) with \( H \) of degree 2. The space of such \( F \) is thus 6 (Exercise: Why?). There are \( \leq n - 5 \) points not on \( G = 0 \). Thus the points on \( L \) have imposed 4 conditions (quadrics are codim 4 in cubics) and the points outside \( L \) impose \( \leq n - 5 \) conditions; hence the total number of conditions imposed by \( \Omega \) is

\[
\leq 4 + n - 5 = n - 1.
\]

Hence if 5 points of \( \Omega \) lie on a line, \( \Omega \) fails to impose independent conditions.

If \( n = 8 \) and \( \Omega \) lies on a conic \( G = 0 \), then the number of conditions imposed by \( \Omega \) on cubic forms is

\[
\text{(dimension of cubics)} - \text{(dimension of cubics } G \cdot H) = 10 - 3 = 7 < 8.
\]

We now prove the \( \Rightarrow \) direction. Assume that \( n \leq 4 \). We can find a cubic form through 3 and not the 4th point for any set of 4 points.
We suppose $n \geq 5$.

Case 1: Suppose $\Omega$ has 4 points on a line $L$ given by $H = 0$. Assume for the moment there are no other points in $\Omega$ on $L$. Let $\Omega'$ be the compliment to the set of 4 points on $L$ in $\Omega$. The set $\Omega'$ has $n - 4 \leq 4$ points since $n \leq 8$.

Claim: $\Omega'$ fails to impose independent conditions on forms of degree 2.
Proof of claim: If $\Omega'$ imposed independent conditions, we could find a form $G$ of degree 2 vanishing at all but one point of $\Omega'$. Then $GH$ would vanish at all but one point of $\Omega$ and hence $\Omega$ would impose independent conditions. This contradicts our hypothesis that $\Omega$ fails to impose independent conditions.

By lemma 2, we conclude that $\#\Omega' = 4$ and the points of $\Omega'$ lie on a line given by $H = 0$. Hence $\Omega$ lies on the conic $GH = 0$. We conclude that in Case 1, either $\Omega$ lies on a conic and $n = 8$ or 5 points of $\Omega$ lie on a line by Lemma 2.

Case 2: Suppose there is a line $L$ containing $l \geq 3$ points of $\Omega$. By the same argument as in Case 1, the remaining $n - l$ points fail to impose independent conditions on curves of degree 2. Therefore these remaining points have 4 among them on a line. This puts us in Case 1.

Case 3: Assume $\Omega$ contains no sets of 3 collinear points. Let $\{P_1, P_2, P_3\}$ be any 3 points in $\Omega$ and let $\Omega'$ be the compliment in $\Omega$ to these 3 points.

- If for any $i, i = 1, 2, 3$, the points $\Omega' \cup \{P_i\}$ impose independent conditions on forms of degree 2, we are done: Let $C$ be a curve of degree 2 containing $\Omega'$ but not $P_i$ and let $L$ be the line through $P_j$ and $P_k$. Then $C \cup L$ is a curve of degree 3 containing all but one point of $\Omega$ and hence $\Omega$ imposes independent conditions. This contradicts our hypothesis.
- Thus we may assume that the points $\Omega' \cup \{P_i\}$ fail to impose independent conditions on forms of degree 2. There are $\leq 6$ points in $\Omega' \cup \{P_i\}$. This set does not contain 4 collinear points (Case 1), and thus, by lemma 2, it contains 6 points that lie on a conic $C_i$. Let $C_j$ and $C_k$ be the conics constructed in the same way for the sets $\Omega' \cup \{P_j\}$ and $\Omega' \cup \{P_k\}$. Now $\Omega'$ consists of 5 points no 3 of which are collinear. There is a unique conic $C$ through the points of $\Omega'$. Thus $C = C_j = C_k = C_k$. Hence $C$ contains all of $\Omega$. □
Proof of Chasles’ theorem: Let $\Gamma'$ be a set of 8 of the points in $C_1 \cap C_2$. We need to show that $\Gamma'$ imposes independent conditions on cubic forms. If $\Gamma' \subseteq D, D$ a conic given by the vanishing of a homogeneous degree 2 form $F$, then $C_i \cap D, i = 1, 2$ has more than the expected number of points, 6. Thus each $C_i = D \cup L_i$ for a line $L_i$. Thus $C_1 \cap C_2$ has more than 9 points (the intersection contains all of $D$). This contradicts our hypothesis, so it is impossible for $\Gamma'$ to be contained in a conic.

If four or more points of $\Gamma'$ were on a line $L$, then using the same argument we see that $L$ would be contained in $C_1$ and $C_2$. As before this leads to a contraction. We conclude that no subset of 4 points of $\Gamma'$ is collinear. By lemma 3 we conclude that $\Gamma'$ imposes independent conditions. \qed