Math 425 Notes 10

Line Integrals

Our goal is to find a method to compute the quantities such as:

- Given a curved wire in space with specified density at each point, find the total mass of the wire.
- Given a portion of surface in space with specified charge density at each point, find the net charge on the whole surface.
- Given part of surface in space find its area.
- Given a region in the plane and a flow on the pane, find the net flow out of the region.
- Given a region in space and a flow in space, find the net flow out of the region.

In each case we reduce the computation to computing an integral on part of \( \mathbb{R} \) or a part of \( \mathbb{R}^2 \) or a part of \( \mathbb{R}^3 \).

Total Mass from Density

We start with a wire in space with given density \( \mu \) that, in general, varies from point to point.

I. We parametrize the curve. This means we find a nice one-to-one and onto map \( \gamma \) from an interval \([a, b]\) of \( \mathbb{R} \) to the curve. Let \( t \) be the coordinate on \( \mathbb{R} \). This is the key step.

II. Look at a very small subinterval of \([a, b]\). Call this \( \Delta t \). It gets mapped to a small portion of the curve. Call this small piece of the curve \( \Delta \gamma \) and let \( p \) be a point on \( \Delta \gamma \). The mass of \( \Delta \gamma \) is approximately

\[
\mu(p) \times (\text{the length of } \Delta \gamma).
\]

III. Let \( \gamma(t_s) = p \). Then the length of \( \Delta \gamma \) is approximately

\[
||\gamma'(t_s)|| \Delta t.
\]

IV. Hence the mass of the curve is approximately

\[
\sum_{i=1}^{i=n} \mu(\gamma(t_i)) ||\gamma'(t_i)|| \Delta t_i
\]

where \( a = t_0 < t_1 < \cdots < t_n = b \) and \( \Delta t_i = t_i - t_{i-1} \).
V. Letting the pieces of the partition of the interval all get smaller and smaller and taking the limit we get that the mass of the curve is

$$\int_{a}^{b} \mu(\gamma(t)) \|\gamma'(t)\| \, dt.$$ 

**Example:** We calculate the total mass of a wire that is the intersection of the sphere of radius 1 with center at the origin and the plane $x + y + z = 0$. The wire has density $y^2$ at the point $(x, y, z)$.

The first step is to parametrize the wire. Since the plane $x + y + z = 0$ passes through the origin the intersection of the plane and the sphere is a great circle of the sphere and has radius 1. Let $f_1, f_2$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^n$ for that matter), then the map

$$\gamma : t \mapsto (\cos(t) f_1, \sin(t) f_2)$$

parametrizes a circle of radius 1 and center the origin contained in the plane spanned by $\{f_1, f_2\}$. We need to find two vectors $\{f_1, f_2\}$ in the plane $x + y = z = 1$ so that $\langle f_i, f_i \rangle = 1$, $\langle f_1, f_2 \rangle = 0$. We find two orthogonal vectors and then we normalize them. I do this in an ad hoc way. Start with

$$f_1^* = (1, 1, -2), \quad f_2^* = (1, -1, 0).$$

Now normalize: $f_1 = \frac{1}{\sqrt{6}} f_1^*$, $f_2 = \frac{1}{\sqrt{2}} f_2^*$. These vectors do the job. The mass of the curve is

$$\int_{0}^{2\pi} y^2 \|\gamma'(t)\| \, dt.$$ 

We need to express the terms in the integral in terms of $t$. We have

$$\gamma'(t) = (-\sin(t)) f_1 + \cos(t) f_2 \implies \|\gamma'(t)\| \equiv 1.$$ 

We have that the $y$-coordinate of $\gamma(t)$ is

$$\cos(t) \frac{1}{\sqrt{6}} + \sin(t) \frac{1}{\sqrt{2}}(-1).$$

Thus we can express all the terms in the integral in terms of $t$ and hence we can integrate it in the usual fashion.

**Work and Force**

Given a force field $F(x, y, z)$ and a path from point $p$ to a point $q$ we show how much work the force field does on a particle moving along the path. We chop the curve into little pieces so that we can use the formula:

$$W = \langle \vec{F}, \vec{s} \rangle$$
where we have an object being displaced by \( \vec{S} \) acted on by a constant force \( \vec{F} \). We denote the points of the partition of the curve by

\[
p = p_0, p_1, \ldots, p_n = q.
\]

The total work done by the vector field is approximately

\[
\sum_{i=1}^{n} < F(P_i), p_i - p_{i-1} >
\]

In computing the mass we needed to know the length of a small piece of the curve, say the length from point \( p_i \) to point \( p_{i+1} \), that is, we needed to know \( ||p_i - p_{i-1}|| \). Here we need more, we need to know the vector \( p_i - p_{i-1} \). Let

\[
\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad a \leq t \leq b,
\]

be a parametrization of our curve and let \( \gamma(t_i) = p_i \). Then

\[
p_i - p_{i-1} \approx D\gamma(t_i)(t_i - t_{i-1}) = \begin{pmatrix} x'(t_i) \\ y'(t_i) \\ z'(t_i) \end{pmatrix} (t_i - t_{i-1}).
\]

If we set \( \Delta t_i = t_i - t_{i-1} \) and \( F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \), the total work done is approximated by

\[
\sum_{i=1}^{n} < F', \gamma'(t) > \Delta t_i = \sum (F_1(x(t_i), y(t_i), z(t_i))x'(t_i) + F_2(x(t_i), y(t_i), z(t_i))y'(t_i) + F_3z'(t_i)) \Delta t_i.
\]

Taking the limit as all \( \Delta t_i \rightarrow 0 \) we get that the total work done is

\[
\int_{t=a}^{t=b} (x'(t)F_1 + y'(t)F_2 + z'(t)F_3) \, dt.
\]

**Example:** Consider the curve \( C \) given by \( y = \frac{1}{2}(e^x + e^{-x}) \) from \( x = -1 \) to \( x = 1 \), and let \( F = (F_1, F_2) = (y, x) \) be a force field. How much work does \( F \) do to a particle moving over \( C \)?

We need to parametrize the curve. We can parametrize any curve given by \( y = f(x) \) by setting

\[
x = t \\
y = f(t).
\]
We need to compute \( \int_{t=1}^{t=-1} (F_1(x(t), y(t))x'(t) + F_2((x(t), y(t))y'(t)) \) dt. We compute

\[
\begin{align*}
  x(t) &= t, & y(t) &= \frac{1}{2}(e^t + e^{-t}), \\
  x'(t) &= 1, & y'(t) &= \frac{1}{2}(e^t - e^{-t}), \\
  F_1(t) &= y(t) = \frac{1}{2}(e^t + e^{-t}), & F_2(t) &= x(t) = t.
\end{align*}
\]

We substitute into our formula and perform the integration. We are done.

Notation: We have integrated \( \int_C (F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt}) \) dt. This is often denoted \( \int_C F_1 dx + F_2 dy \).

**Example:** Let \( c \) be the line from \((1, 1)\) to \((10, 10)\). Let \( F = (3x - 2y, x + y) \). We compute the work down \( f \) on a particle moving over \( c \).

We have that work is

\[
W = \int_C F_1 dx + F_2 dy.
\]

To perform the integration we need to parametrize the curve. In this case we have

\[
\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1 - t) \begin{pmatrix} 10 \\ 10 \end{pmatrix}.
\]

**Flow through a Membrane**

Let \( F \) be the vector field of a fluid flow in the plane that does not vary in time. This means that the velocity of the flow at any point does not change over time. It may change from point to point. We wish to calculate how much fluid flows through a membrane. We start with the case of a vector field that is the same from point to point. Call it \( F = (F_1, F_2) \). Let \( v \) be the vector from point \( p \) to point \( q \). The flow across \( v \) is

\[
\det(F v).
\]

The columns of this matrix are the column vectors \( v \) and \( F \). Note that the determinant is signed. If the determinant is positive it indicates that the flow is from left to right as we look along the vector \( v \). Another way of saying this is: The positive direction of flow is the one gotten by turning \( v \) in the clockwise direction.

We need to generalize this in two directions. We need to work with curves that are not straight line segments. We need to work with a fluid flow that varies from point to point. Both of these problems are solved by chopping up the curve into little pieces. Let \( C \) be a nice curve starting at \( p \) and ending at \( q \). We partition the curve

\[
p = p_0, p_1, \ldots, p_n = q, \quad v_i = p_i - p_{i-1} = \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix}.
\]
The flow out of the \( i \)-th piece is
\[
\det \begin{pmatrix}
F_1(p_i) & \Delta x_i \\
F_2(p_i) & \Delta y_i
\end{pmatrix}.
\]
To get the flow across the curve we add these up and take the limit as the size of the little pieces all go to zero. We can write the result as
\[
\int_C F_1 dy - F_2 dx.
\]

How do we go about computing this? The first step is to parametrize the curve. Say
\[
\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (x(t), y(t))
\]
is a parametrization of \( C \), and \( \gamma(t_i) = p_i \). This allows us to approximate the piece of the curve \( p_{i-1} \) to \( p_i \) by
\[
\begin{pmatrix}
x'(t_i) \\
y'(t_i)
\end{pmatrix}(t_i - t_{i-1}).
\]
We get
\[
\int_a^b (F_1(x(t), y(t))y'(t) - F_2((x(t), y(t))x'(t)) \, dt.
\]
Note that everything is expressed in terms of \( t \).

**Example:** Let \( C \) be the unit circle and let \( F = (x, y) \). We parametrize the circle by
\[
\gamma(t) = (\cos(t), \sin(t)), \quad t \in [0, 2\pi].
\]
This gives
\[
x'(t) = -\sin(t), \quad F_1(x(t), y(t)) = \cos(t) \\
y'(t) = \cos(t), \quad F_2(x(t), y(t)) = \sin(t).
\]
Hence the net flow out of the circle is
\[
\int_0^{2\pi} \left( F_1 \frac{dy}{dt} - F_2 \frac{dx}{dt} \right) \, dt = \int_0^{2\pi} \cos(t)(\cos(t) dt - \sin(t)(-\sin(t))dt = \int_0^{2\pi} (1) \, dt = 2\pi.
\]

We give second way of justifying our formula for the net flux across a curve. We start with a directed line segment from \( p \) to \( q \), so \( v = q - p \). We have a flux \( F = (F_1, F_2) \) that is constant over the plane and ask to calculate the flow of \( F \) across \( v \). Let \( n \) be a vector of unit length that is normal to \( v \). There are two choices. We take the one that rotates \( v \) in a clockwise direction, so if \( v = (v_1, v_2) \), then
\[
n = \frac{(v_2, -v_1)}{\|v\|}.
\]
The flow is
\[
(component \ of \ F \ normal \ to \ v) \cdot (the \ length \ of \ v) = < F, n > \|v\| = < F, v > = F_1v_2 - F_2v_1.
\]
This leads to the same formula we found using the other approach.
Potential Functions and Work

Sometimes a vector field $F$ is a gradient. In that case we call a function $V$ so that $\nabla V = -F$ a potential function for $F$. Note the minus sign. In this case the calculation of work is simplified. Let

$$\gamma : [a, b] \to C \subset \mathbb{R}^3, \ t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

be a parametrized curve $C$ and let $F = (F_1, F_2, F_3)$ be a force field. Then

$$\int_C <F, (dx, dy, dz)> = \int_{a}^{b} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

$$- \int_{a}^{b} \left( - \frac{\partial V}{\partial x} \frac{dx}{dt} - \frac{\partial V}{\partial y} \frac{dy}{dt} - \frac{\partial V}{\partial z} \frac{dz}{dt} \right) dt,$$

but

$$- \frac{\partial V}{\partial x} \frac{dx}{dt} - \frac{\partial V}{\partial y} \frac{dy}{dt} - \frac{\partial V}{\partial z} \frac{dz}{dt}$$

is equal to, by the chain rule,

$$- \frac{dV}{dt}.$$

By the fundamental theorem of calculus the work is

$$V(a) - V(b).$$

Note: In this case the amount of work done does **NOT** depend on the curve between the end points, but rather it just depends on the end points.