Basis of the Kernel

Definition 1. As subset $S$ of $\mathbb{R}^n$ that is closed under addition and scalar multiplication is said to be a subspace of $\mathbb{R}^n$. More concretely $S$ is a subspace provided the two conditions below are satisfied.

- If $s_1, s_2 \in S$, then $s_1 + s_2 \in S$.
- If $s \in S, \lambda \in \mathbb{R}$, then $\lambda s \in S$.

Example 1. Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation. Then $\text{ker}(F)$ and the image of $F$ (which we denote by $\text{im}(F)$) are both subspaces. The kernel of $F$ is a subspace of $\mathbb{R}^m$, the domain, and $\text{im}(F)$ is a subspace of $\mathbb{R}^n$, the target or range of $F$.

Example 2. Let $T$ the subset of $\mathbb{R}^3$ consisting of all the points of $\mathbb{R}^3$ whose coordinates are all integers. Then $T$ is closed under addition, but $T$ is not closed under scalar multiplication. Hence $T$ is not a subspace.

Definition 2. Let $S$ be a subset of $\mathbb{R}^n$. The span of $S$ is the set of all linear combinations of elements of $S$.

Given a linear map we show how to find a small set $S$ of vectors with the property that they span the kernel of our linear map. We do this in an example. Let $A$ be the matrix

$$
\begin{pmatrix}
1 & -1 & 0 & -1 & 1 \\
2 & 0 & 2 & 2 & 4 \\
0 & 1 & 1 & 2 & 1
\end{pmatrix},
$$

so $A$ gives a map from $\mathbb{R}^5$ to $\mathbb{R}^3$. Upon row reduction we get

$$
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Thus the kernel of $A$ is the set of all solutions to the equations

$$
x_1 + x_3 + x_4 + x_5 = 0 \\
x_2 + x_3 + 2x_4 + x_5 = 0.
$$

We can choose the variables $x_3, x_4, x_5$ freely and $x_1, x_2$ are determined by our equations. We have

$$
x_1 = -x_3 + -x_4 + -x_5 \\
x_2 = -x_3 + -2x_4 + -x_5 \\
x_3 = x_3 \\
x_4 = x_4 \\
x_5 = x_5.
$$
We can write this as

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
= \begin{pmatrix}
  -1 \\
  -1 \\
  1 \\
  0 \\
  0
\end{pmatrix} + \begin{pmatrix}
  -1 \\
  -2 \\
  0 \\
  1 \\
  0
\end{pmatrix} + \begin{pmatrix}
  -1 \\
  -1 \\
  0 \\
  0 \\
  1
\end{pmatrix}.
\]

First observe that the three elements

\[
v_1 = \begin{pmatrix}
  -1 \\
  -1 \\
  1 \\
  0 \\
  0
\end{pmatrix}, \quad v_2 = \begin{pmatrix}
  -1 \\
  -2 \\
  0 \\
  1 \\
  0
\end{pmatrix}, \quad v_3 = \begin{pmatrix}
  -1 \\
  -1 \\
  0 \\
  0 \\
  1
\end{pmatrix}
\]

are in the domain of \( A \). Second observe that these three elements are in the kernel of \( F \). We verify this for \( v_1 \) by multiplying

\[
\begin{pmatrix}
  1 & -1 & 0 & -1 & 1 \\
  2 & 0 & 2 & 2 & 4 \\
  0 & 1 & 1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
  -1 \\
  -1 \\
  1 \\
  0 \\
  0
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

We should multiply \( Av_2, Av_3 \) but omit this. We can also verify that we get \( v_1 \) as an element of the kernel by setting \( x_3 = 1, x_4 = 0, x_5 = 0 \) and using the equations above to determine \( x_1, x_2 \).

**Observation 1.** An element of the kernel of \( A \) is determined by the values of \( x_3, x_4, x_5 \).

This says that every element of the kernel of \( A \) is a linear combination of the elements of the set \( B = \{ v_1, v_2, v_3 \} \). This says that \( B \) spans the kernel of \( A \).

**Observation 2.** We can not remove any one of the elements from \( B \) and still span the kernel of \( A \).

Thus we have reached our goal. We have found a **small** set that spans the kernel of \( A \). We now make some definitions about general situation.

**Definition 3.** Let \( B \) be a subset of \( \mathbb{R}^n \). Let \( w \in B \). Let \( B' \) be the set gotten from \( B \) by removing the element \( w \) from \( B \). We say that an element \( w \) of \( B \) is **redundant** provided the span of \( B \) is the same as the span of \( B' \).

If \( w \) is redundant, then it is a linear combination of the elements of \( B \) and hence a linear combination of the elements of \( B' \). One the other hand, if \( w \in B \) is a linear combination of the elements of \( B' \), then the span of \( B \) is the same as the span of \( B' \) and \( w \) is redundant. We have the equivalent definition.
**Definition 4.** Let $B$ be a subset of $\mathbb{R}^n$. Let $w \in B$ and let $B'$ be the same as $B$ except that $w$ has been removed from $B$. We say $w$ is redundant if $w$ is a linear combination of elements of $B'$.

We return to our specific matrix $A$ and the set $B = \{v_1, v_2, v_3\}$. We rephrase our conclusions as the

**Observation 3.** The set $B$ spans the kernel of $A$ and has no redundant elements.

**Definition 5.** Let $V$ be a subspace of $\mathbb{R}^n$. A set $B$ which

- spans the subspace $V$ of $\mathbb{R}^n$ and,
- has no redundant elements

is a **basis** of the subspace $V$.

**Observation 4.** The set $B$ we have calculated is a basis of the kernel of $A$.

**Basis of the Image**

We work an example of how to find the basis of the image of a linear map defined by a matrix. Let $A$ be an $n \times m$ matrix. Then $A$ defines a linear map

$$A : \mathbb{R}^m \rightarrow \mathbb{R}^n.$$

**Lemma 3.** The image of $A$ is spanned by the column vectors of $A$.

**Proof.** Let $e_i$ denote the column vector of length $m$ with all zero entries except in the $i$-th position which has a 1. If we multiply $Ae_i$ we obtain the $i$-th column vector of $A$ which we denote by $A_i$.

We can write an arbitrary vector

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \in \mathbb{R}^m$$

as

$$x_1e_1 + x_2e_2 + \cdots.$$

We have

$$Av = A(x_1e_1 + x_2e_2 + \cdots) = x_1A(e_1) + x_2A(e_2) + \cdots.$$

This exhibits an arbitrary element in the image of $A$ as a linear combination of the column vectors of $A$, that is, as an element in the span of $\{A_1, A_2, \cdots\}$.

**Question 1.** How do we identify redundant elements in the set $C = \{A_1, A_2, A_3, A_4, A_5\}$ of column vectors?
We answer this in an example, namely we identify the redundant vectors among the column vectors of the matrix we used above. Recall that

$$A = \begin{pmatrix}
1 & -1 & 0 & -1 & 1 \\
2 & 0 & 2 & 2 & 4 \\
0 & 1 & 1 & 2 & 1
\end{pmatrix}.$$ 

Recall that a basis of the kernel was the set

$$B = \{v_1 = \begin{pmatrix}
-1 \\
-1 \\
1 \\
0 \\
0
\end{pmatrix}, 
\quad v_2 = \begin{pmatrix}
-1 \\
-2 \\
0 \\
1 \\
0
\end{pmatrix}, 
\quad v_3 = \begin{pmatrix}
-1 \\
-1 \\
0 \\
0 \\
1
\end{pmatrix}\}.$$ 

We put together two facts.

1. The elements $v_1, v_2, v_3$ are in the kernel of $A$, so that $Av_i = 0$, $i = 1, 2, 3$.

2. From the lemma above we have

$$Av_1 = -1A_1 - 1A_2 + 1A_3, \quad Av_2 = -1A_1 - 2A_2 + A_4, \quad Av_3 = -A_1 - A_2 + A_5.$$ 

The first equation says $A_3 = A_1 + A_2$ so that $A_3$ is a redundant element in $C$. Remove $A_3$ from out set, so we are left with $\{A_1, A_2, A_4, A_5\}$. The element $A_4$ is redundant from this smaller set. So we remove that element and we are left with $\{A_1, A_2, A_5\}$. In the same way remove $A_5$. We are left with $\{A_1, A_2\}$. We have used up all the elements in the basis of the kernel, so we stop here. I have not explained why neither of the two remaining elements is redundant. We will do that in the next set of notes.

We conclude that $\{A_1, A_2\}$ is a basis of the image of $A$. 