Definition 1. A vector space is a set $V$ with an operation we call addition and a map that associates to an element $\lambda \in \mathbb{R}$ and an element $v \in V$ another element in $V$ denoted by $\lambda v$. We denote the addition of two elements $v, w$ in $V$ by $v + w$. These operations satisfy

- \( v + w = w + v \).
- \( (u + v) + w = u + (v + w) \).
- There exists an element of $V$, which we denote by $0$, so that $0 + v = v + 0 = v$ for all $v \in V$.
- For all elements $v \in V$ there exists an element $(-v) \in V$ so that $v + (-v) = 0$.
- $1 \cdot v = v$ for all $v \in V$.
- $a(bv) = (ab)v$ for all $a, b \in \mathbb{R}, v \in V$.
- $0 \cdot v = 0$ for all $v \in V$. Here the first $0$ is the zero in $\mathbb{R}$ and the second is the additive identity in $V$.
- $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}, v \in V$.
- $a(v + w) = av + aw$ for all $a \in \mathbb{R}, v, w \in V$.

Example 1. $\mathbb{R}^n$ is a vector space.

Example 2. Let $\mathbb{R}^{n \times m}$ denote the set of all $n \times m$ matrices.

Example 3. Let $P_n$ denote the set of all polynomial functions of degree less than or equal to $n$ in one variable. We define addition as follows: Let $f, g \in P_n$. Then the polynomial $f + g$ is the polynomial

$$ (f + g)(x) = f(x) + g(x). $$

We define scalar multiplication by

$$ (\lambda f)(x) = \lambda \cdot (f(x)). $$

We verify one of the conditions for $P_2$ to be a vector space. Let $f_1 = a_1 + b_1 t + c_1 t^2, f_2 = a_2 + b_2 t + c_2 t^2, a_i, b_i, c_i \in \mathbb{R}$ for $i = 1, 2$. Then by definition

$$ (f_1 + f_2)(t) = f_1(t) + f_2(t) = (a_1 + b_1 t + c_1 t^2) + (a_2 + b_2 t + c_2 t^2) $$
$$ = a_2 + b_2 t + c_2 t^2 + a_1 + b_1 t + c_1 t^2 = (f_2 + f_1)(t). \ \ (1) $$

We have used the commutative of addition of real numbers. The other conditions are similarly reduced to verifying the analogous property for real number arithmetic.
Example 4. Let \( F(\mathbb{R}, \mathbb{R}) \) denote the set of all functions from \( \mathbb{R} \) to \( \mathbb{R} \). We define addition of functions by 
\[
(f + g)(x) = f(x) + g(x).
\]
We define scalar multiplication of a function by 
\[
(\lambda f)(x) = \lambda \cdot f(x), \lambda \in \mathbb{R}.
\]
We verify one property of vector spaces for this example. 
\[
(\lambda(f + g))(x) = \lambda \cdot (f + g)(x) = \lambda \cdot (f(x) + g(x)) = \lambda \cdot f(x) + \lambda \cdot g(x) = (\lambda f)(x) + (\lambda g)(x).
\]

Example 5. Let \( \mathbb{C} \) denote the complex numbers, \( \mathbb{C} = \{x + iy | x, y \in \mathbb{R}\} \). We add complex numbers by 
\[
(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).
\]
We multiply complex numbers by 
\[
(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),
\]
and, in particular, if \( \lambda \in \mathbb{R} \), then 
\[
\lambda \cdot (x + iy) = \lambda x + i\lambda y.
\]
The set of complex numbers forms a vector space.

Subspaces

Definition 2. Let \( V \) be a vector space and let \( W \) be a subset of \( V \). \( W \) is a subspace provided
- If \( W_1, w_2 \in W \), then \( w_1 + w_2 \in W \).
- If \( w \in W, \lambda \in \mathbb{R} \), then \( \lambda w \in W \).

Example 6. Let \( W = \{f \in P_2 | f(3) = 0\} \), then \( W \) is a subspace of \( P_2 \).

Example 7. Let \( W \subset P_2 \) be the set of all elements \( f \) of \( P_2 \) such that \( f(3) = 1 \). Then \( W \) is not a subspace. Here is why. Let \( f, g \in W \). Then 
\[
(f + g)(3) = f(3) + g(3) = 1 + 1 \neq 1.
\]
This means that \( f + g \) is not an element of \( W \).

Example 8. Let \( U \subset \mathbb{R}^{2 \times 2} \) be the matrices all of whose entries below the main diagonal are zero. Then \( U \) is a subspace.

Example 9. Let \( D \subset F(\mathbb{R}, \mathbb{R}) \) be the set of all functions which have a derivative. Then \( D \) is a subspace. We verify that the sum of two elements in \( D \) is still in \( D \). Note that 
\[
(f + g)' = f' + g'.
\]
Inherit in this statement is the fact that the sum of two functions which have derivatives also has a derivative.
Example 10. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Let $W \subset \mathbb{R}^{2 \times 2}$ be the set of all matrices which commute with $A$, that is $W = \{ B | AB = BA \}$. We claim this is a subspace of $\mathbb{R}^{2 \times 2}$.

- If $B$ and $C$ both commute with $A$, then
  
  \[ A(B + C) = AB + AC = BA + CA = (B + C)A. \]

- If $B$ commutes with $A$, then so does $\lambda B$. 
