Let $A$ be the matrix \[
\begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
-3 & 1 & 5
\end{pmatrix}.
\] In this problem you are asked to work out explicitly for the matrix $A$ the construction that is carried out in the proof of the Primary Decomposition Theorem 23.9 page 197 in Curtis’s textbook.

1. Show that the characteristic polynomial of $A$ is $h(x) = (x - 2)^2(x - 5)$.

2. Show that the minimal polynomial $m(x)$ of $A$ is equal to its characteristic polynomial.

3. Set $p_1(x) = (x - 2)$, $e_1 = 2$, $p_2(x) = (x - 5)$, $e_2 = 1$, so that $m(x) = p_1(x)^{e_1} p_2(x)^{e_2}$.

Set $q_1(x) := \frac{m(x)}{p_1(x)^{e_1}} = (x - 5)$, and $q_2(x) := \frac{m(x)}{p_2(x)^{e_2}} = (x - 2)^2$.

Find polynomials $a_1(x)$ and $a_2(x)$ satisfying the equation
\[
1 = a_1(x) q_1(x) + a_2(x) q_2(x).
\] (1)

Hint: You can find $a_1(x) = ax + b$ of degree 1 and $a_2(x) = c$ a constant polynomial. Simply regard the coefficients $a$, $b$, and $c$ as variables, and equate coefficients in the polynomial equation (1) to get three linear equations in three unknowns.

4. Set $f_i(x) = a_i(x) q_i(x)$ and compute
\[
E_i := f_i(A),
\]
$i = 1, 2$. Show that $E_1 E_2 = 0$. Note that $E_1$ and $E_2$ commute, so $E_2 E_1 = 0$ as well, and $E_1 + E_2 = I$, by construction. Hence, $(E_i)^2 = E_i$ and
\[
\mathbb{R}^3 = E_1(\mathbb{R}^3) \oplus E_2(\mathbb{R}^3)
\] (2)
is a direct sum decomposition, by Lemma 23.6.

5. Compute a basis $\{v_1\}$ of $\ker(p_1(A))$. Then extend it to a basis $\{v_1, v_2\}$ for $\ker(p_1(A)^{e_1})$ and show that the latter kernel is equal to $E_1(\mathbb{R}^3)$.

6. Compute a basis $\{v_3\}$ of $\ker(p_2(A)^{e_2})$ and show that the kernel is equal to $E_2(\mathbb{R}^3)$.

7. Observe that the direct sum decomposition displayed in equation (2) is translated via the equalities in parts 5 and 6 to the direct sum decomposition
\[
\mathbb{R}^3 = \ker(p_1(A)^{e_1}) \oplus \ker(p_2(A)^{e_2}).
\]

8. Set $\beta := \{v_1, v_2, v_3\}$. It is a basis of $\mathbb{R}^3$, by the above direct sum decomposition. Compute the $\beta$-matrix $[A]_\beta$ of $A$. It should be block-diagonal with two blocks, each of which is upper triangular.
Answer: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad [A]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

The entry of $[A]_\beta$ in the first row and second column depends on your choice of the basis $\{v_1, v_2\}$. As long as its non-zero, your answer is correct.