Show all your work and justify all your answers!

1. (18 points) Let \( C \) be the circle of radius 2 centered at the origin and oriented counterclockwise. Evaluate the following integrals.
   
   (a) \( \int_C \frac{dz}{z^2 + 2z - 3} \)
   
   (b) \( \int_C \log(z + 5)dz \), where \( \log(z) \) is the principal branch of the logarithm function with argument in \((-\pi, \pi)\).

2. (18 points) Let \( C \) be the unit circle oriented counterclockwise and let \( z_0 \) be a complex number satisfying \( |z_0| \neq 1 \). Prove the equality
   
   \[ \int_C \frac{\sin(z^2)}{(z-z_0)^2}dz = \int_C \frac{2z \cos(z^2)}{z-z_0}dz. \]

3. (10 points) Let \( f(z) = e^{iz^2} \sin(z^4 + z - 2) \). Does \( f \) have an anti-derivative? In other words, does there exist an entire function \( F(z) \), such that \( F'(z) = f(z) \). Carefully justify your answer.

4. (18 points) Let \( C_1 \) be the circle of radius 2 centered at 2i oriented counterclockwise. Let \( C_2 \) be the circle of radius 5 centered at the origin oriented counterclockwise. Set \( f(z) := \frac{1}{(z^2 + 1)^2} \). Evaluate the difference \( \int_{C_2} f(z)dz - \int_{C_1} f(z)dz \).
   
   Hint: Cauchy-Goursat’s Theorem for multiply connected regions helps. Clearly state it and explain why its all hypothesis are satisfied in the set-up in which you apply it..

5. (18 points)
   
   (a) Let \( U \) be the upper half-plane \( \{x+iy : y > 0\} \) of the complex plane. Set \( g(z) := e^{iz} \). Describe geometrically the image \( g(U) \) of \( U \) under the function \( g \).
   
   (b) Suppose that \( f(z) \) is an entire function. Write \( f(x+iy) = u(x,y) + iv(x,y) \). Assume that \( v(x,y) \geq u(x,y) \), for all points \( (x,y) \) in the plane. Note that the assumption means that the values of \( f \) are all in the half-plane above the line \( v = u \) in the \( (u,v) \) plane. Show that \( f(z) \) is a constant function.
   
   Hint: Consider the function \( g(z) = e^{\lambda f(z)} \), for a suitable constant \( \lambda \).

6. (18 points) Let \( C_R \) denote the circle of radius \( R, R > 2 \), centered at the origin and oriented counterclockwise. Set \( I_R := \int_{C_R} \frac{z^2 + 9}{z^4 + 3z^2 + 2}dz \).
   
   (a) Prove the inequality
   
   \[ |I_R| \leq \frac{2\pi R(R^2 + 9)}{(R^2 - 1)(R^2 - 2)}. \]  \( \text{(1)} \)

   (b) Prove that \( \lim_{R \to \infty} I_R = 0 \). Note that you are taking the limit of the \textbf{left} hand side of equation \( (1) \).

   (c) Use part 6b to prove that \( I_R = 0 \), for all \( R \geq 2 \).