Solve problem 1 and only 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____. Show all your work. Credit will not be given for an answer without a justification. Calculators may not be used in this exam.

1. (16 points) Given that the Taylor series of \( \tan(z) \), centered at 0, has the form

\[
\tan(z) = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \cdots \text{ terms of order at least seven.} \tag{1}
\]

a) Evaluate the fifth derivative \( \tan^{(5)}(0) \) with as little calculations as possible.

\[
\tan^{(5)}(0) = \frac{2}{15} \quad \text{so} \quad \tan^{(5)}(0) = \frac{2}{15} \cdot \frac{5!}{5!} = \frac{2 \cdot 5!}{15} = 2 \cdot 4 \cdot 2 = 16
\]

b) Find the principal part at \( z = 0 \) of the function \( f(z) = \frac{(1 + z) \tan(z)}{z^5} \)

\[
\frac{1}{z^5} (1 + z) \left( z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \cdots \right) = \frac{1}{z^4} + \frac{1}{z^3} + \left( \frac{1}{3} \right) z + \left( \frac{1}{3} \right) \frac{1}{z^2} + \left( \frac{1}{3} \right) \frac{1}{z^3} + \cdots
\]

\[
\text{Principal Part}
\]
c) Find all the singularities of \( f(z) \) (given in part b) in the disk \( D = \{ |z| < 4 \} \) and determine their type (isolated, removable, pole of what order, essential).

A pole of order 4 at \( z = 0 \): \( z^2 + 2 \)

\( f(z) = \frac{(1 + z) \sin(z)}{z^5 \cos(z)} \)

A simple pole at \( z = \frac{\pi}{2} \) and \( z = -\frac{\pi}{2} \).

\[ \text{d) Find the residue at each isolated singularity in } D. \]

\[ \text{Res}_{z=0} f(z) = \frac{1}{3} \quad \text{from part (a)} \]

\[ \text{Res}_{z=\frac{\pi}{2}} f(z) = \frac{(1 + \frac{\pi}{2}) \sin(\frac{\pi}{2})}{(\frac{\pi}{2})^5} = - \frac{(1 + \frac{\pi}{2})}{(\frac{\pi}{2})^5} \]

\[ \text{Res}_{z=-\frac{\pi}{2}} f(z) = \frac{(1 - \frac{\pi}{2}) \sin(-\frac{\pi}{2})}{(-\frac{\pi}{2})^5} = \frac{(1 - \frac{\pi}{2})}{(-\frac{\pi}{2})^5} \]
2. (12 points) a) Compute \( \sin(\pi + i \ln(3)) \). Simplify your answer as much as possible.

\[
\begin{align*}
\frac{e^{i(\pi + i \ln(3))} - e^{-i(\ln(3) + \pi i)}}{2i} &= \frac{e^{-\ln(3) - \pi i} - e^{-\ln(3) + \pi i}}{2i} \\
&= \frac{(-1)^{\ln(3) + \pi}}{2i} \\
&= (-1)^{\ln(3)/3 - 3/2i} = \left(\frac{1}{3} - 3/2i\right) = -\frac{4}{3}i
\end{align*}
\]

8 points

b) Prove that all solutions of the equation \( \cos(z) = 0 \) are real and find all the solutions.

\[
\frac{e^{iz} - e^{-iz}}{2i} = 0
\]

\[
e^{iz} = -e^{-iz}
\]

\[
e^{2iz} = -1 = e^{\pi i}
\]

\[
2iz = i[\pi + 2\pi n]
\]

\[
z = \frac{\pi}{2} + 2\pi n
\]
3. (12 points) Compute the integral \( \int_C \frac{z^5}{1-z^3} \, dz \), where \( C \) is the circle of radius 2, centered at 0, and traversed counterclockwise.

\[
\int_C \frac{z^5}{1-z^3} \, dz = 2\pi i \left[ \frac{z^5}{1-z^3} \right]_{z=1}^{z=e^{2\pi i/3}} + \frac{1}{-3(i^3)}
\]

\[
\left[ \frac{z^5}{1-z^3} \right]_{z=1}^{z=e^{2\pi i/3}} = \frac{(e^{2\pi i/3})^5}{1-(e^{2\pi i/3})^3} = \frac{1}{-3(e^{2\pi i/3})^2} = \frac{1}{-3 \cdot 2}
\]

\[
2\pi i \left[ -\frac{1}{3} \right] = -\frac{2\pi i}{3}
\]

\( z^3 = 1 \)
4. (12 points) a) Find the Taylor series of the function \( f(z) = \frac{2z + 1}{z^2 + z - 2} = \frac{1}{z - 1} + \frac{1}{z + 2} \) centered at 0 and determine its radius of convergence. Justify your answer.

\[
\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{m=0}^{\infty} z^m
\]
\[
\frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1 + \frac{z}{2}} = \frac{1}{2} \sum_{m=0}^{\infty} (-1)^m 2^{-m} z^m = \sum_{m=0}^{\infty} (-1)^m 2^{-m-1} z^m
\]

\[
\theta(z) = \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} (-1)^r 2^{-r-1} z^m
\]

The largest disk centered at 0 which does not contain \( 1 - 2 \).

b) Find the Laurent series of the function \( f(z) \), given in part a), valid in the annulus \( 1 < |z| < 2 \).

\[
\frac{1}{z-1} \text{ is analytic in } |z| > 1 \Rightarrow \frac{1}{z-1} = \frac{1}{z} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} z^{2m-1}
\]
\[
\frac{1}{z+2} \text{ is analytic in } |z| < 2
\]
\[
\sum_{r=0}^{\infty} z^m + \sum_{m=0}^{\infty} (-1)^m 2^{-m-1} z^m
\]
\[
\sum_{m=-\infty}^{\infty} z^m
\]
5. (12 points) a) Use the definition of contour integrals to prove the equality

\[ \int_C \sin(\bar{z}) \, dz = \int_C \sin(1/z) \, dz, \]  

where \( C \) is the circle \( \{ z : |z| = 1 \} \), traversed counterclockwise. Caution: The argument of the integrand, on the left hand side, is the complex conjugate \( \bar{z} \) of \( z \).

b) Find the Laurent series of \( \sin(1/z) \) centered at zero and classify the type of singularity at \( z = 0 \).

\[
\sin(1/z) = \frac{1}{z} - \frac{1}{(3!) z^3} + \frac{1}{(5!) z^5} + \ldots + \frac{(-1)^n}{(2n+1)!} \, \frac{1}{z^{2n+1}} + \ldots
\]

The origin is an **ESSENTIAL** singularity.

c) Use the equality (2) in order to evaluate the integral \( \int_C \sin(z) \, dz \).

\[
\int_C \sin(z) \, dz = \int_C \sin(1/z) \, dz = 2\pi i \ \text{Res} \ \sin(1/z) = \frac{\sin(1/2)}{2} = (2\pi i) \cdot \frac{1}{2} = \frac{2\pi i}{2} = \pi i.
\]
6. (12 points) Evaluate the integral

\[ I = \int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta)}. \]

Show all your work!

\[ Z = e^{i\theta} = \cos(\theta) + i\sin(\theta) \]

\[ \frac{1}{Z} = e^{-i\theta} = \cos(\theta) - i\sin(\theta) \]

\[ \cos(\theta) = \frac{(Z + \frac{1}{Z})}{2i} \]

\[ dZ = i e^{i\theta} d\theta = iZd\theta \]

\[ d\theta = \frac{dZ}{Z} \]

\[ I = \int \frac{dZ}{Z \left[ a + \frac{Z + \frac{1}{Z}}{2} \right]} = -2i \int \frac{dZ}{1Z^2 + 2Z + 1} \]

\[ a = -2 \pm \sqrt{4 - 1} = -2 \pm \sqrt{1} = -2 \pm 1 \]

\[ a = -2 + \sqrt{3}, \quad \text{Re} \left\{ \frac{1}{Z^2 + 4Z + 1} \right\} \]

\[ Z = -2 + \sqrt{3} \]

\[ \frac{1}{\left( -2 + \sqrt{3} \right) + 4} \]

\[ = \frac{4\pi}{\sqrt{3}} \cdot \frac{1}{2(-2 + \sqrt{3}) + 4} \]

\[ = \frac{4\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}} \]
7. (12 points) Let \( S_R \) be the upper-semi-circle of radius \( R > 1 \), given by the parametrization \( z = Re^{i\theta}, 0 \leq \theta \leq \pi \). Prove the equality

\[
\lim_{R \to \infty} \int_{S_R} \frac{z^2 dz}{1 + z^4} = 0.
\]

*Hint:* Find first an upper bound for the integral.
8. (12 points) Determine whether the following statements are true or false. Justify your answers!

a) Let $C$ be the circle $\{z : |z| = 1\}$, oriented counterclockwise. Assume that $f(z)$ is analytic in the punctured disk $0 < |z| < 2$, and the integrals $\int_C z^n f(z) \, dz$ vanish, for all integers $n \geq 0$. Then 0 is a removable singularity of $f$.

True,

The Laurent series of $f$ in the given punctured disk is
$$\sum_{k=-\infty}^{\infty} c_k z^k$$
where
$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{k+1}} \, dz.$$ Setting $k = -m-1$, we see that $c_k = 0$ for $k \leq -1$, and so the principal part of the Laurent series vanishes.

b) There exists a function $F(z)$, analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$, whose derivative $f(z) := F'(z)$ satisfies $\text{Res}_{z=0}(f(z)) = 1$.

False.

$\text{Res}_{z=0} f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} \, dz$, where $C$ is a circle of radius $r$, $0 < r < 1$, centered at 0. The function $f$ can not have an anti-derivative in $U$, since its contour integral along the closed curve $C'$ in $U$ is non-zero.
c) If \( f \) is a non-constant entire function and \( |f(z)| \leq 2 \), for every \( z \) on the unit circle \( \{z : |z| = 1\} \), then \( f \) must map the unit disk \( \{z : |z| < 1\} \) into the disk \( \{z : |z| < 2\} \).

True, by the Maximum modulus principle.

Since if \( f \) is non constant, and \( |f(z)| \leq 2 \) for \( |z| = 1 \) then \( |f(z)| < 2 \) for all \( |z| < 1 \).

d) There exists an entire function, whose real part is \( e^{x+y} \).

False.

The real part of an analytic function is a Harmonic function.

Set \( u(x,y) = e^{x+y} \),

Then \( u_{xx} + u_{yy} = 2e^{x+y} \neq 0 \),

So \( u \) is not harmonic.
9. (12 points) Evaluate the improper integral

\[ \int_0^\infty \frac{x^2}{x^4 + 1} \, dx. \]

\[ \deg (x^2) = 2 \]
\[ \deg (x^4 + 1) = 4 \geq 2 + 2. \text{ So} \]
\[ \int_0^\infty \frac{x^2}{x^4 + 1} \, dx = \frac{1}{2} \sum_{\text{residues in upper-half plane}} \left( \frac{z^2}{z^4 + 1} \right) \]
\[ = \frac{2 \pi i}{2} \left[ \text{Res} \left( \frac{z^2}{z^4 + 1} \right)_{z = \frac{\pi i}{2}} + \text{Res} \left( \frac{z^2}{z^4 + 1} \right)_{z = e^{\pi i/4}} \right] \]
\[ = \frac{\pi i}{4} \cdot \left[ e^{-\pi/4} + e^{-3\pi i/4} \right] = \frac{\sqrt{2} \pi}{4} = \frac{\pi}{2\sqrt{2}} \]

\[ \left. \left( \frac{z^2}{z^4 + 1} \right) \right|_{z = e^{\pi i/4}} = \frac{3 \pi i}{4} \]
\[ \left. \left( \frac{z^2}{z^4 + 1} \right) \right|_{z = e^{-\pi i/4}} = \frac{-3 \pi i}{4} \]
\[ \frac{1}{4} e^{\pi i/4} = \frac{e^{-\pi i/4}}{4} \]

\[ e^{3\pi i/4} \]