There are totally 9 problems in this exam.

Show all your work !!!

1. (15 pts) Let P, Q be statements. Show that the statement NOT (P OR Q) is equivalent to the statement (NOT P) AND (NOT Q) by showing they have the same truth tables.

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<th>~P V ~Q</th>
<th>(~P) ∧ (~Q)</th>
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2. (10 pts) Prove or give a counterexample to the following statement:

\( \forall x \in \mathbb{Z}, \ 2x^3 + 2x + 5 \neq 0. \)

Proof by contradiction. Suppose there is an integer solution \( x_0 \in \mathbb{Z}. \) Then \( 2x_0^3 + 2x_0 + 5 = 0 \)

This implies \( 2(x_0^3 + 0) = -5 \)

which is contradiction because LHS is even but RHS = -5 which is odd.
3. (10 pts) Prove, using the contrapositive method, that
if \(2x^3 + 3x^2 + 5x - 10 \leq 0\), then \(x \leq 1\).

By contrapositive method, we assume \(x > 1\).
Then \(2x^3 + 3x^2 + 5x - 10 > 2 \cdot 1^3 + 3 \cdot 1^2 + 5 \cdot 1 - 10\)
\[= 2 + 3 + 5 - 10 = 0\]
This is true because for \(x > 1\), \(x^3, x^2, x\) are increasing.

Hence if \(x > 1\), then \(2x^3 + 3x^2 + 5x - 10 > 0\), which,
by contrapositive method, shows that
if \(2x^3 + 3x^2 + 5x - 10 \leq 0\), then \(x \leq 1\).

4. (10 pts) Define the sequence \(x_n\) as follows:
\[x_1 = x_2 = 6, \text{ and for } n \geq 3, \ x_n = 2x_{n-1} + 3x_{n-2}.\]
Prove that for all \(n \geq 1\), \(x_n = 3(3^{n-1} + (-1)^{n-1})\).

Prove by Math Induction.

Step 1: Verify for \(n = 1, 2\). For \(n = 1\), LHS = \(x_1 = 6\)
RHS = \(3 \cdot (3^{1-1} + (-1)^{1-1}) = 3 \cdot (1 + 1) = 6\)
For \(n = 2\), LHS = \(x_2 = 6\), RHS = \(3 \cdot (3^{2-1} + (-1)^{2-1}) = 3 \cdot (3 + 1)\)
\[= 6\]

Step 2: Assume \(x_{k+1} = 3(3^{k-1} + (-1)^{k-1})\)
\(x_k = 3(3^{k-1} + (-1)^{k-1})\).

Then \(x_{k+1} = 2x_k + 3x_{k-1} = 2 \cdot 3(3^{k-1} + (-1)^{k-1}) + 3 \cdot 3(3^{k-2} + (-1)^{k-2})\)
\[= 6(3^{k-1} + (-1)^{k-1}) + 9(3^{k-2} + (-1)^{k-2})\]
\[= 2 \cdot 3^k - 6 \cdot (-1)^k + 9 \cdot 3^k + 9 \cdot (-1)^k\]
\[= 3 \cdot 3^k + 3 \cdot (-1)^k = 3(3^k + (-1)^k).\]
5. (10 pts) Determine whether the following is true or not, and explain why.

\[
\text{NOT} \ (\forall x \in D, P(x) \Rightarrow Q(x)) \text{ is equivalent to } \exists x \in D, ((\text{NOT} \ P(x)) \text{ AND } Q(x)).
\]

They are not equivalent.

Explanation: 
\[
\sim (\sim (\forall x \in D, P(x) \Rightarrow Q(x))) = \forall x \in D, P(x) \Rightarrow Q(x),
\]
\[
\sim (\exists x \in D, \sim (P(x) \lor Q(x))) = \forall x \in D, \sim (\sim P(x) \land Q(x)) = \forall x \in D, (Q(x) \Rightarrow P(x))
\]
\[
(\forall x \ P(x) \Rightarrow Q(x)) \neq (\forall x \in D, \sim Q(x))
\]

6. (10 pts) Find an expression for 

\[
S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n, \text{ where } n = 1, 2, 3, \ldots,
\]

and prove the expression for \( S_n \) is correct.

Suppose \( n = 2k \) is even, then

\[
S_n = S_{2k} = 1 - 2 + 3 - 4 + \cdots + \frac{(-1)^{2k+1}}{2k} 2k
\]

\[
= (1 - 2) + (3 - 4) + \cdots + \left[\frac{2k - 1}{2k} - \frac{2k}{2k}\right]
\]

\[
= (-1) + (-1) + \cdots + (-1) = k \cdot (-1) = -k
\]

When \( n = 2k+1 \) odd, then

\[
S_n = S_{2k+1} = 1 - 2 + 3 - 4 + \cdots + \frac{(-1)^{2k+1+1}}{2k+1} (2k+1)
\]

\[
= (1 - 2) + (3 - 4) + \cdots + \left[\frac{2k - 1}{2k} - \frac{2k}{2k}\right] + (2k + 1)
\]

\[
= (-1) + (-1) + \cdots + (-1) + (2k + 1) = -k + 2k + 1
\]

\[
= k + 1.
\]

So \( S_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \)
7. (15 pts) Consider the linear Diophantine equation: \(10x + 25y = 200\).

(1) Find all integer solutions of the equation.

(2) Find all non-negative integer solutions of the equation.

Step 1: Solve \(10x + 25y = 5\). By guessing, \(x = -2, y = 1\) is a solution. From this we obtain one particular solution \((x_0, y_0)\) of \(10x + 25y = 200\), where

\[
\begin{align*}
    x_0 &= x \cdot \frac{200}{5} = (-2) \cdot \frac{200}{5} = -80 \\
    y_0 &= y \cdot \frac{200}{5} = 1 \cdot \frac{200}{5} = 40
\end{align*}
\]

Step 2: Find all solutions:

\[
\begin{align*}
    x &= x_0 + \frac{-b}{\text{gcd} m = -80 + \frac{-25}{5} m = -80 - 5m} \\
    y &= y_0 + \frac{a}{\text{gcd} m = 40 + \frac{10}{5} m = 40 + 2m, m \in \mathbb{Z}}
\end{align*}
\]

Step 3: Find all non-negative solutions:

\[
\begin{align*}
    x \geq 0 &\implies -80 - 5m \geq 0 \implies 20 \leq m \leq -16 \\
    y \geq 0 &\implies 40 + 2m \geq 0
\end{align*}
\]

So \(m = -16, -17, -18, -19, -20\), and 

\[
\begin{align*}
    (x, y) &= (0, 8), (5, 6), (10, 4), \\
    &\quad (15, 2), (20, 0)
\end{align*}
\]
8. (10 pts) Determine whether 223 is prime or not, and explain why.

\[ \sqrt{223} < 15. \] So we check whether p is a divisor of 223 for \( p = 2, 3, 5, 7, 11, 13 \).

\[ 2 | 223, \quad 3 | 223, \quad 5 | 223, \quad 7 | 223, \]

\[ 11 | 223, \quad 13 | 223. \] Hence 223 is prime.

9. (10 pts) Let \( u, v \in \mathbb{Z} \). Suppose \( \gcd(3, u) = 1 \). Prove that if \( u \mid 9v \), then \( u \mid v \).

Since \( q = 3^2 \), \( \gcd(3, u) = 1 \) \( \Rightarrow \) \( \gcd(q, u) = 1 \).

Now there are \( x, y \in \mathbb{Z} \), such that

\[ qx + uy = 1. \]

Then \( N = v \cdot 1 = v \cdot (qx + uy) = x \cdot (9v) + (uv) \cdot u \) which means that \( v \) is a linear combination of \( 9v \) and \( u \). Since \( u \mid 9v \) and \( u \mid u \), we have \( u \mid v \).