Double Induction:

Let \( f(k, m) \) be a function of two positive integers, defined recursively by:

(a) \( f(1, 1) = 2 \)

(b) \( f(k+1, m) = f(k, m) + 2(k+m) \)

(c) \( f(k, m+1) = f(k, m) + 2(k+m-1) \)

for all \( k, m \geq 1 \). Prove that

\[
f(k, m) = k^2 + 2km + m^2 - k - 3m + 2
\]

Proof:

Step 1: We will prove, by induction on \( m \), that

\[
f(1, m) = 1^2 + 2m + m^2 - 1 - 3m + 2 = m^2 - m + 2
\]

(a) \( f(1, 1) = 2 \), \( 1^2 - 1 + 2 = 2 \).

(c) Assume that \( f(1, m) = m^2 - m + 2 \). We need to show

\[
f(1, m+1) = f(1, m) + 2m = m^2 - m + 2 + 2m = m^2 + m + 2
\]

Ind. Hyp. \( (m+1)^2 - (m+1) + 2 \).

Hence \( f(1, m) = m^2 - m + 2 \), for all \( m \in \mathbb{N} \).

Step 2: Consider the statement \( P(k) := \]

\[
f(k, m) = k^2 + 2km + m^2 - k - 3m + 2, \text{ for all } m
\]

We will prove it, for all \( k \), by induction on \( k \).

(1) The case \( k = 1 \) is step 1.
(2i) Assume \( P(k) \).

We need to prove \( P(k+1) \), which states

\[
\beta(k+1) = (k+1)^2 + 2(k+1)m + m^2 - (k+1)^2 - 3m + 2
\]

\[
= k^2 + 2k + 1 + 2km + 2m + m^2 - k - 1 - 3m + 2
\]

\[
\beta(k+1) = \beta(k,m) + 2k + 2m \uparrow
\]

\( \text{Ind. Hyp.} \)

\[
= (k^2 + 2km + m^2 - k - 3m + 2) + 2k + 2m
\]

Hence \( P(k+1) \) holds as well.

We conclude that statement \( P(k) \) holds for all positive integers \( k \), by the principle of induction.

Hence, equality (\( \Box \)) holds for all positive integers \( k \) and \( m \).

Q.E.D