MATH 300  FINAL EXAM  NAME:

There are 10 problems in this exam.

Show all your work. Good luck !!!

1. This problem consists of 3 independent parts.
   (1) (5 pts) Negate the expression: \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R}, (x^2 - y = 1) \Rightarrow (x \neq y) \).
   (2) (5 pts) Disprove: \( \forall x \in \mathbb{R}, \) if \( x^3 + 40x^2 - 41 \geq 0 \), then \( x > 1 \).
   (3) (5 pts) Let \( R \) be the relation on the set \( \mathbb{Q} \) defined as follows: for any \( x, y \in \mathbb{Q} \), \( xRy \) if \( x - y \geq 0 \). Is \( R \) an equivalence relation? Why?
2. (10 pts) Prove that $x^3 + 2x - 2 = 0$ has no integer solutions by contradiction.

3. (10 pts) Solve for $x \in \mathbb{Z}$: $2x^{33} + 5x + 1 \equiv 0 \pmod{34}$. 
4. (10 pts) Find the last 2 digits of $423^{324}$.

5. (10 pts) Solve linear congruence: $48x \equiv 20 \pmod{44}$.
6. (10 pts) Prove that there exists no \( m \in \mathbb{Z} \) such that \( 7m + 6 \) is a perfect square.

7. (10 pts) Show that the function \( f : \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{11} \), where \( f([x]) = [6] \cdot [x] \), is a bijection. Furthermore, find a formula for the inverse \( f^{-1} : \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{11} \).
8. (5 pts) Use Mathematical Induction to show that for any $n \geq 1$,

$$1 + t + t^2 + \cdots + t^n = \frac{1 - t^{n+1}}{1 - t}.$$
9. (10 pts) Solve for $x \in \mathbb{Z}: x^3 \equiv 11 \pmod{30}$.

10. (10 pts) Prove that $\# \mathbb{P} = \#(\mathbb{P}_2 \times \mathbb{P})$ by constructing an explicit bijection (i.e., given by a formula) from $\mathbb{P}$ to $\mathbb{P}_2 \times \mathbb{P}$. Furthermore, find a formula for the inverse of your bijection.