1. (15 points) The matrices $A$ and $B$ below are row equivalent (you do not need to check this fact).

$$A = \begin{pmatrix}
1 & -3 & 4 & -1 & 9 \\
-2 & 6 & -6 & -1 & -10 \\
-3 & 9 & -6 & -6 & -3 \\
3 & -9 & 4 & 9 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & -3 & 0 & 5 & -7 \\
0 & 0 & 2 & -3 & 8 \\
0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

a) What is the rank of $A$?

b) Find a basis for the null space $Null(A)$ of $A$.

c) Find a basis for the column space of $A$.

d) Find a basis for the row space of $A$.

2. (4 points) The null space of the $5 \times 6$ matrix $A$ is 2 dimensional. What is the dimension of (a) the Row space of $A$? (b) the Column space of $A$? Justify your answer!

3. (15 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
4 & -4 & -1
\end{pmatrix}$

is $-(\lambda - 1)(\lambda + 1)(\lambda - 2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that the matrix $A$ above satisfies

$$P^{-1}AP = D$$

4. (12 points) Determine for which of the following matrices $A$ below there exists an invertible matrix $P$ (with real entries) such that $P^{-1}AP$ is a diagonal matrix. You do not need to find $P$. Justify your answer!

(a) \( \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \)

(b) \( \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} \)

(c) \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

5. (22 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Note: Parts 5a, 5b, 5c are mutually independent and are not needed for doing parts 5d, 5e, 5f.
(a) Find the length of $v_1$.

(b) Find the distance between the two points $v_1$ and $v_2$ in $\mathbb{R}^3$.

(c) Find a vector of length 1 which is orthogonal to $W$.

(d) Find the projection of $v_2$ to the line spanned by $v_1$.

(e) Write $v_2$ as the sum of a vector parallel to $v_1$ and a vector orthogonal to $v_1$.

(f) Find an orthogonal basis for $W$.

6. (16 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(a) Find the projection of $b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ to $W$.

(b) Find the distance from $b$ to $W$.

(c) Find a least square solution to the equation $Ax = b$ where $A$ is the $3 \times 2$ matrix with columns $u_1$ and $u_2$. I.e., find a vector $x$ in $\mathbb{R}^2$ which minimizes the length $\|Ax - b\|$. 

(d) Find the coefficients $c_0, c_1$ of the line $y(x) = c_0 + c_1x$ which best fits the three points $(x_1, y_1) = (-1, 0), (x_2, y_2) = (0, 2), (x_3, y_3) = (1, 1)$ in the $x, y$ plane.

The line should minimize the sum $\sum_{i=1}^{3} [y(x_i) - y_i]^2$. **Justify your answer!**

7. (16 points) The vectors $v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .8 & .5 \\ .2 & .5 \end{pmatrix}$.

(a) The eigenvalue of $v_1$ is ______

The eigenvalue of $v_2$ is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(d) As $n$ gets larger, the vector $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ approaches _____. Justify your answer.