1. (18 points)

(a) Consider the complex plane \( \mathbb{C} \) as a two dimensional vector space with basis \( \beta = \{1, i\} \). Let \( T : \mathbb{C} \rightarrow \mathbb{C} \) be multiplication by the complex number \( 2 + 3i \), i.e., \( T(z) = (2 + 3i)z \). Find the \( \beta \)-matrix of \( T \).

(b) Let \( A = \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix} \). Find the characteristic polynomial of \( A \) and determine the eigenvalues of \( A \).

(c) Find an invertible matrix \( P \), with complex entries, and a diagonal matrix \( D \), such that \( P^{-1}AP = D \). Justify your answer!

(d) Find an invertible matrix \( S \), with real entries, and real numbers \( a, b \), such that \( S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \). Justify your answer.

2. (18 points)

(a) Assume given a \( 3 \times 3 \) matrix \( A \) and a \( 3 \times 3 \) upper triangular matrix \( U = \begin{pmatrix} 2 & u_{12} & u_{13} \\ 0 & 3 & u_{23} \\ 0 & 0 & 5 \end{pmatrix} \). Consider the sequence of row operations

1) Interchange row 1 and row 2 of \( A \) to obtain the matrix \( B \).
2) Multiply by \( \frac{1}{2} \) row 3 of \( B \) to obtain the matrix \( C \).
3) Add \(-2\) times row 1 to row 2 of \( C \) to obtain the matrix \( D \).
4) Add \(-3\) times row 2 to row 3 of \( E \) to obtain the matrix \( U \).
Assume that these elementary row operations reduce \( A \) to \( U \). Compute \( \det(A) \). Justify your answer!

(b) For which values of the real constants \( a \) and \( b \) is the matrix \( \begin{pmatrix} 2 & a \\ 0 & b \end{pmatrix} \) diagonalizable? Justify your answer!

(c) Let \( \mathbb{R}^{3\times3} \) be the vector space of matrices of size \( 3 \times 3 \) and \( T : \mathbb{R}^{3\times3} \rightarrow \mathbb{R}^4 \) a linear transformation. What are all the possible values of \( \dim(\ker(T)) \)? Justify your answer!

3. (a) (5 points) Find all orthogonal matrices of the form \( \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & b \\ \frac{1}{\sqrt{3}} & 0 & c \end{pmatrix} \).

(b) (5 points) Let \( A \) be an \( n \times n \) matrix and \( A^T \) its transpose. Recall that \( \det(A) = \det(A^T) \) and \( \det(AB) = \det(A) \det(B) \) for any \( n \times n \) matrix \( B \). Use the above properties of the determinant to show that if \( A \) is an orthogonal \( n \times n \) matrix, then \( \det(A) \) is equal to 1 or \(-1\).
4. (18 points) The vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$.

(a) The eigenvalue of $v_1$ is _______.
(b) Set $w := \begin{pmatrix} 13 \\ 13 \end{pmatrix}$. Find the coordinate vector $[w]_\beta$ of $w$ in the basis $\beta := \{v_1, v_2\}$.
(c) Compute $A^{100} \begin{pmatrix} 13 \\ 13 \end{pmatrix}$.
(d) As $n$ gets larger, the vector $A^n \begin{pmatrix} 13 \\ 13 \end{pmatrix}$ approaches _______. Justify your answer.

5. (18 points)

(a) Let $P$ be the vector space of polynomials of arbitrary degree. Consider the transformation $T : P \to P$, given by $T(f(t)) = t^2 f''(t) - 2tf(t) + 2f''(t)$. Show that $T$ is linear.
(b) $P_2$ the subspace of $P$ of polynomials of degree $\leq 2$. Note that $T$ maps $P_2$ into $P_2$. Let $S : P_2 \to P_2$ be given by the same formula above, $S(f(t)) = t^2 f'(t) - 2tf(t) + 2f'(t)$. Find the matrix of $S$ in the basis $\beta = \{1, t, t^2\}$.
(c) Determine if $S$ is an isomorphism. Justify your answer!
(d) The function $f(t) = t^2 - 2t + 2$ is an eigenvector of $S$. What is its eigenvalue? Justify your answer!

6. (18 points) Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$, and $V$ the subspace of $\mathbb{R}^4$ spanned by $v_1$ and $v_2$.

(a) Let $w = \begin{pmatrix} 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Find the orthogonal projection $\text{Proj}_V(w)$ of $w$ to $V$. Justify your answer!
(b) Write $w$ as a sum of a vector in $V$ and a vector orthogonal to $V$.
(c) Find the distance from $w$ to $V$, i.e., the distance from $w$ to the vector in $V$ closest to $w$.
(d) Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the set $\beta := \{v_1, v_2, w\}$. Use the Gram-Schmidt process with the basis $\beta$ of $W$ to find an orthonormal basis of $W$. Explain every step of the Gram-Schmidt process you used.