Instructions:

- Please use correct notation when writing matrices and vectors.
- In True-or-False questions, please give reasoning or a counter-example. Examples alone are not reasons.
- Explain how you arrived at your answers, please, and show your algebraic calculations. Use the back of the preceding page if necessary.
- Please justify your statements. Unsubstantiated answers receive no credit.
- \textbf{bf} If \( L \) is a linear map from a vector space \( V \) to \( V \) and \( B \) is a basis of \( V \), then we denote the matrix of \( L \) with respect to the basis \( B \) as \( [L]_{BB} \). In some sections slightly different notations were used for this.

1: True or False. (Please support your answer with a brief reason or a counter-example.)
1a: Suppose \( L : \mathbb{R}^n \to \mathbb{R}^n \) is linear. Assume that for all \( y \in \mathbb{R}^n \) the equation \( Lx = y \) has a solution. Then the solution is unique.
1b: Let \( P_n \) denote the vector space of polynomials of degree less than or equal to \( n \). Let \( F : P_3 \to P_2 \) be linear. Then \( F(p) = 2 - 3t^2 \) has a solution.
1c: Assume that \( S = \{u_1, u_2, u_3\} \) is a set of three non-zero orthogonal vectors in \( \mathbb{R}^3 \), so \( <u_i, u_j> = 0 \) if \( i \neq j \). Then \( S \) is a basis of \( \mathbb{R}^3 \).
1d: Suppose that \( L : \mathbb{R}^n \to \mathbb{R}^m \) is linear and that \( x_1, x_2 \) are both solutions to the equation \( Lx = y \). Then \( x_1 - x_2 \) is in the kernel of \( L \).
1e: Suppose that \( P \) is the vector space of all polynomials in the variable \( t \). Let
\[
T : P \to P, \quad p(t) \mapsto \frac{d^2p}{dt^2} - 2\frac{dp}{dt} - 3p + 7.
\]
Then \( T \) is linear.

2: Let \( P_2 \) be the vector space of polynomials of degree less than or equal to \( 2 \) with basis \( S = \{1, t, t^2\} \). Let \( T : P_2 \to P_2 : p(t) \mapsto 3p''(t) - 2p'(t) + p(t) \).
2a: Verify that \( T \) is a linear map.
2b: Compute the matrix of \( T \) with respect to the basis \( S \).
2c: Is \( T \) an isomorphism? Why?
2d: Find all polynomials \( p(t) \) so that \( T(p(t)) = 2 - 3t^2 \).
3: Let \( S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \) be the standard basis, and let \( B = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \) be another basis of \( \mathbb{R}^2 \).

3a: Suppose \( v \in \mathbb{R}^2 \) has coordinates \([v]_S = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\) with respect to the standard basis \( S \). What are its coordinates \([v]_B\) with respect to \( B \)?

3b: If a linear map \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) has matrix
\[
[L]_S = \begin{pmatrix} 9 & -8 \\ 10 & -9 \end{pmatrix}
\]
in the standard basis \( S \), what is its matrix \([L]_B\) in the basis \( B \)?

4: Compute the determinant of the matrix
\[
B = \begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & 3 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}.
\]
Find \( \det(B^{23}) \). Find \( \det(B^{-1}) \) if \( B \) is invertible.

5: Let \( X = \begin{pmatrix} 1 & -4 \\ -16 & -11 \end{pmatrix} \).

5a: Compute the characteristic polynomial of \( X \).
5b: Find the eigenvalues of \( X \).
5c: Find the eigenvectors for each eigenvalue.
5d: Is \( X \) diagonalizable, that is, can we find a matrix so that \( AXA^{-1} \) is a diagonal matrix? If so find the matrix \( A \). If not explain why not.

6: Let \( M = \begin{pmatrix} 5 & -5 \\ 2 & -1 \end{pmatrix} \).

6a: Compute the characteristic polynomial of \( M \).
6b: Find the eigenvalues and eigenvectors of \( M \).
6c: The eigenvalues are not real. Find a basis \( B \) of \( \mathbb{R}^2 \) so that the matrix of \( M \) with respect to the basis \( B \) is of the form
\[
\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
\]

7: Let \( P_2 \) denote the vector space of polynomials of degree less than or equal to 2. Is \( S = \{1, 1-t, 1-t^2\} \) a basis of \( P_2 \)? Why? (Note that explaining why is the important part of the question.)