You are given below the matrix $A$ together with its row reduced echelon form $B$

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 4 \\ 2 & 0 & 4 & 3 & 5 & 2 \\ 3 & 2 & 4 & 6 & 9 & 10 \end{pmatrix}$$

You do not need to check that $A$ and $B$ are indeed row equivalent.

(a) Find a basis for the kernel $\ker(A)$ of $A$.

**Solution:** The vectors

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

are linearly independent vectors in the kernel of $A$. Since the nullity of $A$ is 3, the vectors form a basis for the kernel of $A$.

(b) Find a basis for the image $\text{im}(A)$ of $A$.

**Solution:** The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 0 \end{pmatrix},$$

form a basis for the image of $A$ since the 3rd, 5th and 6th columns of $A$ are redundant vectors among the columns of $A$.

(c) Does the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ belong to the image of $A$? Use part (b) to minimize your computations. Justify your answer!

**Solution:** Yes, since

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 6 \end{pmatrix}.$$

(2) (12 points)

(a) Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation. What are the possible values of $\dim(\ker(T))$? Justify your answer!

**Answer:** The equality $\dim(\ker(T)) + \dim(\text{im}(T)) = 7$ holds, by the Rank-Nullity Theorem. The inequality $\dim(\text{im}(T)) \leq 4$ holds, since $\text{im}(T)$ is a subspace of $\mathbb{R}^4$. Hence, $\dim(\ker(T)) = 7 - \dim(\text{im}(T)) \geq 7 - 4 = 3$. The inequality $\dim(\ker(T)) \leq 7$ holds, since $\ker(T)$ is a subspace of $\mathbb{R}^7$. We conclude that $3 \leq \dim(\ker(T)) \leq 7$.

(b) Let $A$ and $B$ be $n \times n$ matrices. Assume that $AB = 0$. Show that the image of $B$ is contained in the kernel of $A$.

**Answer:** Let $\vec{y}$ be a vector in the image of $B$. Then $\vec{y} = B\vec{x}$, for some $\vec{x}$ in $\mathbb{R}^n$, by definition of the image of $B$. The vector $\vec{y}$ is in the kernel of $A$, if $A\vec{y} = \vec{0}$. The latter is indeed the case, since we have

$$A\vec{y} = A(B\vec{x}) = (AB)\vec{x} = \vec{0},$$
where the rightmost equality follows from the assumption that $AB = 0$.

(c) Let $A$ and $B$ be $n \times n$ matrices and assume that the image of $B$ is contained in the kernel of $A$. Show that $\text{rank}(B) \leq \dim(\ker(A))$. Explain why it follows that $\text{rank}(A) + \text{rank}(B) \leq n$.

**Answer:** The equality $\text{rank}(B) = \dim(\text{im}(B))$ is the definition of $\text{rank}(B)$. The inequality $\dim(\text{im}(B)) \leq \dim(\ker(A))$ holds, since $\text{im}(B)$ is assumed a subspace of $\ker(A)$. We conclude the inequality $\text{rank}(A) + \text{rank}(B) \leq \text{rank}(A) + \dim(\ker(A))$. Now the right hand side is $n$, by the Rank-Nullity Theorem. We conclude the inequality $\text{rank}(A) + \text{rank}(B) \leq n$.

(3) (18 points) Let $\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(a) Show that $\{\vec{v}_2, \vec{v}_3\}$ form a basis for the subspace $P$ of $\mathbb{R}^3$ orthogonal to $\vec{v}_1$.

**Answer:** We have that $P = \ker \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. Since $P$ is the kernel of a $1 \times 3$ matrix of rank 1, it is a vector space (Theorem 3.2.2) of dimension $\dim P = 3 - 1 = 2$ (by the rank-nullity theorem). Geometrically, $P$ is a plane through the origin in $\mathbb{R}^3$ with normal vector $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We have

$$v_1 \cdot v_2 = \frac{1}{\sqrt{3}}(1 - 1 + 0) = v_1 \cdot v_3 = 0.$$ 

Hence $v_2 \in P$ and $v_3 \in P$. They are linearly independent, since the 3-rd entry of $v_2$ is 0 and the 3-rd entry of $v_3$ is $-1 \neq 0$. (Theorem 3.2.5, p.117). Since $\dim P = 2$, $\text{span}\{v_2, v_3\} = P$.

(b) Consider the basis $\beta := \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_1$. Find the $\beta$-matrix $B$ of $T$ (the matrix of $T$ in the basis $\beta$). Justify your answer!

**Answer:** By Theorem 4.3.2, p.174 (or Definition 3.4.3, p.143) we have

$$B = [T(v_1)_\beta \ T(v_2)_\beta \ T(v_3)_\beta].$$

By part a), $P = \text{span}\{v_2, v_3\}$ and $v_3$ is orthogonal to $P$. The linear transformation $T$ is a reflection with respect to the plane $P$, hence

$$T(v_1) = -v_1, \quad T(v_2) = v_2, \quad T(v_3) = v_3,$$

and thus

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

(c) Let $S$ be the $3 \times 3$ matrix $(\vec{v}_1 \vec{v}_2 \vec{v}_3)$ with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$. Express the standard matrix $A$ of $T$ in terms of the matrix $S$ and the matrix $B$ you found in part (b). (You do not need to simplify your answer).

**Answer:** We have the following commutative diagramme (pp.145, 174)

(see Definition 4.1.3 for $L_\beta$.) Thus

$$A = SBS^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1}.$$ 

(4) (18 points) Let $\mathbb{R}^{2 \times 2}$ be the vector space of $2 \times 2$ matrices and $P$ an invertible $2 \times 2$ matrix. Let $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ be the function sending a matrix $M$ to $T(M) = P^{-1}MP$. 

(a) Show that $T$ is a linear transformation.

**Sketch of answer:** One shows that $T(M + N) = T(M) + T(N)$ and $T(kM) = kT(M)$.

(b) Show that $T$ is an isomorphism by explicitly finding $T^{-1}$. Carefully justify your answer!

**Sketch of answer:** We have $T^{-1}(M) = PMP^{-1}$ because $T(T^{-1}(M)) = M$.

(c) Assume now that $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find the matrix $B$ of $T$ in part (c) in the basis $\beta := \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ of $\mathbb{R}^{2 \times 2}$.

**Sketch of answer:** First find $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ to compute $T(e_1) = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$, and $T(e_4) = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$. Then the matrix is

$$B = \begin{pmatrix} 2 & 1 & -2 & -1 \\ 1 & 1 & -1 & -1 \\ -2 & -1 & 4 & 2 \\ -1 & -1 & 2 & 2 \end{pmatrix}.$$

(5) (16 points) Let $P_2$ be the vector space of polynomials of degree $\leq 2$. (i) Which of the following subsets $W$ of $P_2$ are subspaces? In each case verify the three conditions in the definition of a subspace, or demonstrate that one of them is violated.

(ii) Find a basis for those that are subspaces.

(a) $W = \{ f(t) : f'(0) = 1 \}$ is the subset of polynomial functions $f(t)$, such that the value of its derivative at $t = 0$ is $1$.

(b) $W = \{ f(t) : f(1) = f'(2) \}$.

**Answer:**

(a) i) $W$ is not a subspace, since it does not contain the zero polynomial. (If $f(t) = 0$, then $f'(0) = 0 \neq 1$).

ii) not applicable.

(b) i) $W$ is a subspace because

   (i) It contains the zero polynomial (If $f(t) = 0$, then $f(1) = f'(2) = 0$).

   (ii) It’s closed under addition: If $f(t)$ and $g(t)$ are in $W$, then $(f + g)(1) = f(1) + g(1) = f'(2) + g'(2) = (f + g)'(2)$.

   (iii) It’s closed under scalar multiplication: If $f(t)$ is in $W$ and $k$ is a scalar, then $(kf)(1) = k \cdot f(1) = k \cdot f'(2) = (kf)'(2)$.

ii) If $f(t) = a + bt + ct^2$ then $f(1) = a + b + c$ and $f'(2) = b + 4c$, so $f$ is in $W$ if and only if $a + b + c = b + 4c$.

We see $b$ can be anything and $a = 3c$. So a general element of $W$ is of the form $3c + bt + ct^2 = b(t) + c(3 + t^2)$ and a basis of $W$ is $\{ t, 3 + t^2 \}$.

(6) (18 points) Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation given by $T(f(t)) = \begin{bmatrix} f'(0) \\ f(1) \\ f(-1) \end{bmatrix}$.

The first entry on the right hand side above is the value of the derivative $f'$ at $0$.

(a) Find a basis (consisting of polynomials) for the kernel $\ker(T)$. Carefully justify why the set you found is a basis.

**Solution:** Suppose $f(t) = a + bt + ct^2$ is in $\ker(T)$. Then $T(f(t)) = 0$, i.e. $\begin{bmatrix} f'(0) \\ f(1) \\ f(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In other words $\begin{bmatrix} b \\ a + b + c \\ a - b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving these three equations, we see that
$b = 0$ and $a = -c$, so $f$ is of the form $f(t) = -c + ct^2 = c(-1 + t^2)$. This shows that every polynomial in $\ker(T)$ is a scalar multiple of $-1 + t^2$, so $\{-1 + t^2\}$ is a basis for $\ker(T)$.

(b) Use your answer in part (a) in order to determine the rank and nullity of $T$. Justify your answer!

**Solution:** From part (a), $\text{nullity}(T) = \dim(\ker(T)) = 1$. By the rank-nullity theorem, $\dim(P_2) = \text{rank}(T) + \text{nullity}(T)$, i.e. $3 = \text{rank}(T) + 1$, so $\text{rank}(T) = 2$.

(c) Find a basis for the image $\text{im}(T)$. Justify your answer!

**Solution:** If $\vec{v}$ is in $\text{im}(T)$, then $\vec{v} = T(a + bt + ct^2)$ for some polynomial $a + bt + ct^2$. This means

$$\vec{v} = T(a + bt + ct^2) = \begin{bmatrix} b \\ a + b + c \\ a - b + c \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (a + c) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$  

So every element of the image of $T$ is a linear combination of the linearly independent vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so a basis for $\text{im}(T)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. 
