(1) (16 points)

a) Show that the reduced row echelon form of the augmented matrix of the system

\[
\begin{align*}
  x_1 + x_2 + 2x_4 + x_5 &= 3 \\
  x_1 - x_3 + x_4 + x_5 &= 2 \\
  -2x_1 + 2x_3 - 2x_4 - x_5 &= -3
\end{align*}
\]

is

\[
\begin{pmatrix}
  1 & 0 & -1 & 1 & 0 & 1 \\
  0 & 1 & 1 & 1 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\].

Use at most six elementary row operations. (Partial credit will be given if you use more). Clearly write in words each elementary row operation you use.

Solution:

\[
\begin{pmatrix}
  1 & 0 & -1 & 1 & 0 & 1 \\
  1 & 0 & -1 & 1 & 1 & 2 \\
  -2 & 0 & 2 & -2 & -1 & -3 \\
\end{pmatrix}
\]

1. Subtract row 1 from row 2
2. Add twice row 1 to row 3

\[
\begin{pmatrix}
  1 & 1 & 0 & 2 & 1 & 3 \\
  0 & -1 & -1 & 0 & -1 \\
  0 & 2 & 2 & 2 & 1 & 3 \\
\end{pmatrix}
\]

3. Multiply row 2 by \((-1)\)

\[
\begin{pmatrix}
  1 & 1 & 0 & 2 & 1 & 3 \\
  0 & 1 & 1 & 1 & 0 & 1 \\
  0 & 2 & 2 & 2 & 1 & 3 \\
\end{pmatrix}
\]

4. Subtract row 2 from row 1
5. Subtract twice row 2 from row 3

\[
\begin{pmatrix}
  1 & 0 & -1 & 1 & 1 & 2 \\
  0 & 1 & 1 & 1 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

6. Subtract row 3 from row 1

\[
\begin{pmatrix}
  1 & 0 & -1 & 1 & 0 & 1 \\
  0 & 1 & 1 & 1 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

b) Find the general solution of the system.

Solution: From the reduced row echelon form of the system, we can see that \(x_3\) and \(x_4\) are the free variables. Letting \(x_3 = s\) and \(x_4 = t\), we have the general solution:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix}
= \begin{bmatrix}
  s - t + 1 \\
  -s - t + 1 \\
  s \\
  t \\
  1 \\
\end{bmatrix}
\]

for all \(s, t \in \mathbb{R}\).
(2) (16 points) Let $A$ be a $5 \times 3$ matrix (5 rows and 3 columns), $\vec{b}, \vec{c}, \vec{d}$ three vectors in $\mathbb{R}^5$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, with variables $x_1, x_2, x_3$. You are told that the matrix equation $A\vec{x} = \vec{b}$ has a unique solution. Carefully justify using complete sentences your answers to the following questions.

(a) What is the row reduced echelon form of $A$?

**Solution:** Since the matrix equation $A\vec{x} = \vec{b}$ has a unique solution, the matrix $A$ must have rank equal to the number of columns. Thus

$$
\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

In particular, since the rank of $A$ is 3, there is at most one solution to any system of the form $A\vec{x} = \vec{v}$.

(b) What can you say about the number of solutions of the system $A\vec{x} = \vec{0}$?

**Solution:** The system is consistent, and has the unique solution given by the zero vector $\vec{0}$ in $\mathbb{R}^3$.

(c) You are given the additional information that the system $A\vec{x} = \vec{c}$ is consistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{b} + \vec{c}$?

**Solution:** Suppose that $\vec{x}_b$ and $\vec{x}_c$ are the unique solutions to the systems $A\vec{x} = \vec{b}$ and that $A\vec{x} = \vec{c}$ respectively. Then

$$
A(\vec{x}_b + \vec{x}_c) = A\vec{x}_b + A\vec{x}_c = \vec{b} + \vec{c}
$$

and so there is one unique solution to the system $A\vec{x} = \vec{b} + \vec{c}$.

(d) What can you say about the number of solutions of the system $A\vec{x} = \vec{d}$?

**Solution:** Since the rank of $A$ is 3, there is at most one solution to the problem.
(3) (18 points) You can solve parts b and c below even without solving part a.

a) Let $L$ be the line in $\mathbb{R}^2$ through the origin and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by the formula

$$Ref_L(\vec{x}) = \frac{2(\vec{x} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})} \vec{v} - \vec{x},$$

where $\vec{x} \cdot \vec{v}$ is the dot product of $\vec{x}$ and $\vec{v}$. Use the above formula to find the matrix $A$ of $Ref_L$, so that $Ref_L(\vec{x}) = A\vec{x}$, for all vectors $\vec{x}$ in $\mathbb{R}^2$. Credit will not be given for an answer which does not derive the entries of $A$ from equation (1) above.

**Solution:** We compute

$$Ref_L(\vec{x}) = \frac{2(\vec{x} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})} \vec{v} - \vec{x}$$

$$= 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2 \left(\frac{3x_1 + 2x_2}{9 + 4}\right) \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 6x_1 + 4x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 18x_1 + 12x_2 \\ 12x_1 + 8x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 5x_1 + 12x_2 \\ 12x_1 - 5x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 5 \\ 12 \end{bmatrix} \vec{x}.$$ 

Thus $A = \frac{1}{13} \begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$. 

b) Let \( \theta \) be the angle from the \( x_1 \)-axis in \( \mathbb{R}^2 \) to the line \( L \) in part a. Denote by \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) the rotation of the plane an angle \( \theta \) counterclockwise about the origin. Note that \( T \) maps the \( x_1 \)-axis onto \( L \) and the \( x_2 \)-axis onto the line perpendicular to \( L \). Use geometric considerations, justified via both sketches and complete sentences, in order to compute the following:

**Solution:** Consider the following sketch:

\[ T(e_1) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \]

i) \( \text{Ref}_L(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \)

**Solution:** The transformation \( T \) sends the vector \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) to the vector \( T(e_1) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \). Since \( T(e_1) \) is parallel to line \( L \), its reflection across \( L \) is \( T(e_1) \).

ii) \( \text{Ref}_L(T\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \)

**Solution:** The transformation \( T \) sends the vector \( e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) to the vector \( T(e_2) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \). Since \( T(e_2) \) is perpendicular to \( L \), its reflection across \( L \) is \(-T(e_2)\).

c) Let \( B \) be the matrix of \( T \) in part b. Use your work in part b to prove the equality \( AB = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} \).

**Solution:** We know that the first column of the matrix \( AB = \text{Ref}_L(T(e_1)) \) and that the second column of \( AB = \text{Ref}_L(T(e_2)) \). Since \( \theta \) is the angle between the \( x_1 \)-axis and the line \( L \), we have \( \cos \theta = \frac{3}{\sqrt{13}} \) and \( \sin \theta = \frac{2}{\sqrt{13}} \), which verifies the equality.
Find all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ that commute with the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, i.e., which satisfy $AM = MA$.

Follow the following three steps.

a) Translate the equation (2) to a system of linear equations that the variables $w$, $x$, $y$, and $z$ should satisfy, in order for $M$ and $A$ to commute.

**Solution:** We write out explicitly the equation $AM = MA$:

\[
\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}
\]

\[
\begin{align*}
2y &= x \\
2z &= 2w + 3x \\
w + 3y &= z \\
x + 3z &= 2y + 3z
\end{align*}
\]

Here we discarded the fourth equation: after cancelling $3z$ it reduces to the first one.

b) Find the general solution $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ of the system in part a.

**Solution:** At this point you can follow one of two routes.

**Method 1: Gauss Elimination**

The system we want to solve is homogeneous and has coefficient matrix

\[
\begin{bmatrix} 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & -2 \\ 1 & 0 & 3 & -1 \end{bmatrix}
\]

Swapping the first and third rows we obtain

\[
\begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 3 & 0 & -2 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 3 & -6 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Here the first operation is $R_2 \leftrightarrow R_2 - 2R_1$. For the second step we observe that $R_2$ and $R_3$ are multiples (if you prefer, $R_2 \leftrightarrow \frac{1}{3}R_2$, $R_3 \leftrightarrow R_3 - R_2$). Then the general solution is

\[
\begin{align*}
w &= t - 3s \\
x &= 2s \\
y &= s \\
z &= t, \quad s, t \in \mathbb{R}.
\end{align*}
\]
\[
\begin{bmatrix}
w \\
x \\
y \\
\end{bmatrix} = s \begin{bmatrix}
-3 \\
2 \\
1 \\
\end{bmatrix} + t \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}, s, t \in \mathbb{R}
\]

**Method 2**

You can solve the system
\[
\begin{align*}
x - 2y &= 0 \\
2w + 3x - 2z &= 0 \\
w + 3y - z &= 0
\end{align*}
\]
“by hand”. First we observe that the second equation can be eliminated: if \( x = 2y \) and \( w = z - 3y \), then \( 2w + 3x - z = 6z - 6y + 6y - 2z = 0 \). Thus we have to solve
\[
\begin{align*}
x - 2y &= 0 \\
w + 3y - z &= 0
\end{align*}
\]

*Note:* We did the same two operations as in the row-reduction above!

There are many (five) ways to choose two of the variables as independent (Gauss elimination would have made this choice for you!). One popular choice was to take \( w \) and \( y \) as parameters, so \( x = 2y \) and \( z = w + 3y \), that is:
\[
\begin{align*}
w &= t \\
x &= 2r \\
y &= r \\
z &= t + 3r
\end{align*}
\]
that is,
\[
\begin{bmatrix}
w \\
x \\
y \\
z \\
\end{bmatrix} = r \begin{bmatrix}
0 \\
2 \\
1 \\
3 \\
\end{bmatrix} + t \begin{bmatrix}
1 \\
0 \\
0 \\
1 \\
\end{bmatrix}, t, r \in \mathbb{R}.
\]

*Note: Method 1 and Method 2 give the same set of solutions!*

c) Find the general form of a matrix \( M \), which commutes with \( A \).

**Solution:**

Here we just copy our result in matrix form. That is,
\[
M = \begin{bmatrix}
w & x \\
y & z \\
\end{bmatrix} = \begin{bmatrix}
t - 3s & 2s \\
\end{bmatrix} =
\]
\[
= s \begin{bmatrix}
-3 & 2 \\
1 & 0 \\
\end{bmatrix} + t \begin{bmatrix}
1 \\
0 \\
0 \\
1 \\
\end{bmatrix}, s, t \in \mathbb{R}.
\]

If you did not use Gauss elimination but followed **Method 2**, you result would look like
\[
M = \begin{bmatrix}
w & x \\
y & z \\
\end{bmatrix} = \begin{bmatrix}
t & 2r \\
\end{bmatrix} =
\]
\[
= t \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} + r \begin{bmatrix}
0 & 1 \\
2 & 3 \\
\end{bmatrix} = tI_2 + rA, t, r \in \mathbb{R}.
\]

*Observe:*

Our general solution is a linear combination of two natural candidates: the
identity matrix $I_2$ and the matrix $A$ itself! It is clear that both $I_2$ and $A$ commute with $A$, and so does any linear combination of theirs. Interestingly, this gives us all matrices commuting with $A$. But then, since $A^2 = AA$ commutes with $A$ (why?), $A^2$ must be a linear combination of $I_2$ and $A$. Consequently, any power $A^n$, $n \geq 2$ is a combination of $I_2$ and $A$. 
(5) (a) (7 points) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. Compute $A^{-1}$. Show all your work.

**Answer:** Row reduce $(A|I) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}$. Hence, $A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \end{pmatrix}$.

(b) (9 points) Determine which of the following linear transformations $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ are invertible. Give a reason, if it is not invertible. If the inverse exists describe it geometrically.

(i) $T$ is the rotation of $\mathbb{R}^2$ 45 degrees counterclockwise.

**Answer:** $T$ is invertible. Its inverse is the rotation of the plane 45 degrees clockwise.

(ii) $T$ is the reflection of $\mathbb{R}^2$ with respect to a line $L$ through the origin and a non-zero vector $u = (u_1, u_2)$.

**Answer:** $T$ is invertible. $T$ is its own inverse, $T^{-1} = T$.

(iii) $T$ is the projection of $\mathbb{R}^2$ onto the line $L$ in part 5(b)ii.

**Answer:** $T$ is not invertible. A function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible, if the equation $T(\vec{x}) = \vec{y}$ has a unique solution $\vec{x}$, for every $\vec{y} \in \mathbb{R}^2$. If $\vec{y}$ does not belong to the line $L$, the equation does not have any solution. If $\vec{y}$ belongs to $L$, the equation has infinitely many solutions (all vectors on the line through $\vec{y}$ orthogonal to $L$).
(6) (18 points)

a) Let \( T : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \) be the linear transformation \( T(\vec{x}) = A\vec{x} \), where
\[
A = \begin{pmatrix}
1 & 2 & 0 & 3 & 1 \\
2 & 4 & -1 & 2 & 2 \\
1 & 2 & 1 & 7 & 2
\end{pmatrix}.
\]
You are given that \( A \) is row equivalent to the matrix \( B = \begin{pmatrix}
1 & 2 & 0 & 3 & 1 \\
0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \).

You do not need to verify this fact. Find a basis for the kernel of \( T \). In other words, find a set of vectors which span \( \ker(T) \) and which is linearly independent.

**Solution:** Let \((x_1, x_2, x_3, x_4, x_5)\) be the coordinates for \( \mathbb{R}^5 \). Then by just solving the corresponding linear system equal to zero, one gets as general solution
\[x_2(-2, 1, 0, 0, 0) + x_4(-3, 0, -4, 1, 0).\]
So, these two vectors span the kernel. They are linearly independent because clearly one is not scalar multiple of the other.

b) Let \( L \) be the line in \( \mathbb{R}^3 \) spanned by the vector \( \vec{v} := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \). Denote by \( L^\perp \) the set of all vectors \( \vec{x} \) in \( \mathbb{R}^3 \) that are orthogonal to \( L \) (i.e., to \( \vec{v} \)). So \( L^\perp \) consists of all vectors \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \), such that the dot product \( \vec{v} \cdot \vec{x} = 0 \) is zero. Show that \( L^\perp \) is a subspace of \( \mathbb{R}^3 \) by stating the three properties defining a subspace and verifying that \( L^\perp \) satisfies each of them.

**Solution:** One checks the three properties. Let \( \vec{x}, \vec{y} \) be orthogonal to \( \vec{v} \) and let \( t \) be a number, then (i) zero vector is there because \( \vec{0} \cdot \vec{v} = 0 \). (ii) the sum is there because \( \vec{v} \cdot (\vec{x} + \vec{y}) = \vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y} = 0 + 0 = 0 \). (iii) scalar multiplication is there because \( \vec{v} \cdot (t\vec{x}) = t(\vec{v} \cdot \vec{x}) = 0 \).