1. (15 points) a) Show that the row reduced echelon form of the augmented matrix
of the system
\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + 3x_5 &= 1 \\
2x_1 + x_2 + x_4 + 4x_5 &= 1 \\
x_1 - x_3 + x_4 + 2x_5 &= 0
\end{align*}
\]
is
\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\].
Use at most seven elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

b) Find the general solution for the system.
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
=
\]

2. a) (8 points) Find the inverse of the matrix
\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{pmatrix}
\].

b) (2 points) Use matrix multiplication to check that the matrix you found is indeed \(A^{-1}\).

c) (5 points) Let \(A, B, C\) be \(n \times n\) matrices, with \(A\) and \(B\) invertible, which satisfy the equation \(ABC^{-1} - B = A\). Express \(C\) in terms of \(A\) and \(B\). Show all your work.

3. (18 points) Recall that two \(n \times n\) matrices \(A\) and \(B\) are said to commute, if \(AB = BA\).

(a) Find all \(2 \times 2\) matrices, which commute with the matrix
\[
A = \begin{pmatrix}
2 & 0 \\
3 & 2
\end{pmatrix}
\].

(b) Let \(A\) and \(B\) be two \(n \times n\) matrices. Show that if \(A\) commutes with \(B\) and \(B\) is invertible, then \(A\) commutes with \(B^{-1}\).

4. (17 points) Let \(A\) be an \(m \times n\) matrix, \(\vec{b}\) a non-zero vector in \(\mathbb{R}^n\), \(\vec{x}_1\) a solution of the equation \(A\vec{x} = \vec{b}\), and \(\vec{x}_h\) a solution of the equation \(A\vec{x} = \vec{0}\).

(a) Show that \(\vec{x}_1 + \vec{x}_h\) is a solution of the equation \(A\vec{x} = \vec{b}\).

(b) Let \(\vec{x}_2\) be another solution of the system \(A\vec{x} = \vec{b}\). Show that \(\vec{x}_2 - \vec{x}_1\) is a solution of the system \(A\vec{x} = \vec{0}\).

(c) Let \(A\) be the \(2 \times 2\) matrix of the projection of \(\mathbb{R}^2\) onto a line \(L\) through the origin and a non-zero vector \(\vec{b}\). Let \(\vec{u}\) be a unit vector orthogonal to \(L\). Draw a picture describing geometrically the set of solutions \(\vec{x}\) of the system \(A\vec{x} = \vec{b}\), in terms of \(\vec{u}\) and \(\vec{b}\). Then use your work in parts 4a and 4b to justify the picture in a paragraph consisting of complete sentences.
5. (20 points) Let \( L \) be the line in \( \mathbb{R}^2 \) through the origin and the vector \( \vec{v} = \left( \frac{1}{\sqrt{3}} \right) \).

Recall that the reflection \( \text{Ref}_L : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by the formula

\[
\text{Ref}_L(x) = \frac{2(x \cdot \vec{v})}{\vec{v} \cdot \vec{v}} \vec{v} - x. \quad (1)
\]

(a) Use the formula (1) to find the standard matrix \( A \) of \( \text{Ref}_L \).

(b) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the rotation of the plane about the origin \( \frac{\pi}{3} \) radians (i.e., 60 degrees) counter-clockwise. Find the standard matrix \( B \) of the rotation \( T \). Hint: \( \cos(\pi/3) = 1/2 \) and \( \sin(\pi/3) = \sqrt{3}/2 \).

(c) Let \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation given by \( S(x) = \text{Ref}_L(T(x)) \) (i.e., rotation followed by reflection). Express the standard matrix \( C \) of \( S \) in terms of the matrices \( A \) of \( \text{Ref}_L \) and \( B \) of \( T \).

\[ C = \text{___________} \]

(d) Use the expression in part 5c to compute the matrix \( C \). Note: The answer is

\[
C = \left( \begin{array}{cc}
1/2 & \sqrt{3}/2 \\
\sqrt{3}/2 & -1/2
\end{array} \right)
\]

(e) Let \( \tilde{L} \) be the line through the origin and the vector \( \tilde{w} = \left( \frac{\sqrt{3}}{1} \right) \). The matrix \( C \) in part 5d is the matrix of the reflection \( \text{Ref}_{\tilde{L}} \) with respect to this new line \( \tilde{L} \). You need not prove this fact. Use this fact and your work above in order to express the rotation \( T \) in terms of the reflections \( \text{Ref}_L \) and \( \text{Ref}_{\tilde{L}} \).

\[ T(x) = \text{________________________} \]. Justify your answer!

6. (15 points)

(a) Is the vector \( \left( \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \right) \) a linear combination of the vectors \( \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \) and \( \left( \begin{array}{c} 4 \\ 3 \\ 2 \end{array} \right) \)?

Justify your answer!

(b) Let \( A \) be a \( 4 \times 3 \) matrix such that the system \( A\vec{x} = \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \) has a unique solution.

i. What is the rank of \( A \)? Justify your answer!

ii. Let \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) be the linear transformation given by \( T(\vec{x}) = A\vec{x} \). Is the image of \( T \) equals the whole of \( \mathbb{R}^4 \)? Justify your answer!