1. (16 points) The matrices $A$ and $B$ below are row equivalent (you do not need to check this fact).

\[ A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

a) Find a basis for the null space $\text{Null}(A)$ of $A$.

b) Find a basis for the column space of $A$.

c) Find a basis for the row space of $A$.

2. (16 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 6 & 0 & -4 \\ 0 & 1 & 0 \\ 8 & 0 & -6 \end{pmatrix}$ is $- (\lambda - 1)(\lambda + 2)(\lambda - 2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that the matrix $A$ above satisfies $P^{-1}AP = D$.

3. (4 point) Let $A$ be a $6 \times 10$ matrix (6 rows and 10 columns). Denote the dimension of the column space of $A$ by $r$.

(a) The dimension $r$ of the column space must be in the range $\underline{\phantom{0}} \leq r \leq \underline{\phantom{0}}$.

(b) Express the dimension of the null space of $A$ in terms of $r$.

\[ \dim \text{Null}(A) = \underline{\phantom{0}} \]

(c) Express the dimension of the row space of $A$ in terms of $r$.

\[ \dim \text{Row}(A) = \underline{\phantom{0}} \]

4. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

(a) The eigenvalue of $v_1$ is $\underline{\phantom{0}}$

The eigenvalue of $v_2$ is $\underline{\phantom{0}}$

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) As $n$ gets larger, the vector $A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches $\underline{\phantom{0}}$. Justify your answer.
5. (16 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -7 \\ 4 \\ -4 \end{bmatrix}$.

(a) Find the projection $\text{Proj}_W(v)$ of $v = \begin{bmatrix} 8 \\ 7 \\ -7 \end{bmatrix}$ to $W$.

(b) Find the distance from $v$ to $W$.

(c) Set $u_3 := v - \text{Proj}_W(v)$ and let $U$ be the $3 \times 3$ matrix with columns $u_1$, $u_2$, and $u_3$. Show that $\frac{1}{9}U$ is an orthogonal matrix.

(d) Find the distance, from the vector $\left(\frac{1}{9}U^T\right)v$ to the plane in $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, without any further calculations. Explain your answer!

Hint: where does $\frac{1}{9}U$ take the three vectors above?

(e) $\frac{1}{9}U$ is the matrix of a rotation of $\mathbb{R}^3$ about a line $L$ through the origin (you may assume this fact). Find a vector $w$ which spans the line $L$ (the axis of rotation).

6. (16 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by $a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(a) Find the projection of $a_2$ to the line spanned by $a_1$.

(b) Find the distance from $a_2$ to the line spanned by $a_1$.

(c) Use your calculations in parts 6a and 6b to show that the vectors $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form an orthogonal basis of the plane $W$ given above.

(d) Find the projection of $b = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ to $W$.

(e) Find a least square solution of the equation $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the $3 \times 2$ matrix with columns $a_1$ and $a_2$. I.e., find a vector $x$ in $\mathbb{R}^2$, for which the distance $\|Ax - b\|$ from $Ax$ to $b$ is minimal.

7. (16 points)

(a) Find the matrix $A$ of the rotation of $\mathbb{R}^2$ an angle of $\frac{\pi}{4}$ radians (45°) counter-clockwise.

(b) Find the matrix $B$ of the reflection of the plane about the line $x_2 = 0$ (the $x_1$ coordinate line).

(c) Compute $C = B^{-1}AB$.

(d) Show that $C$ is the matrix of a rotation and find the angle of rotation.