1. (15 points) The matrices $A$ and $B$ below are row equivalent (you do not need to check this fact).

\[
A = \begin{pmatrix}
1 & -2 & 1 & 1 & 1 \\
2 & -4 & 0 & 1 & 3 \\
-3 & 6 & 1 & 1 & -3 \\
-1 & 2 & 0 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & -2 & 1 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

a) What is the rank of $A$?

b) Find a basis for the null space $\text{Null}(A)$ of $A$.

c) Find a basis for the column space of $A$.

d) Find a basis for the row space of $A$.

2. (6 points) The system $A\vec{x} = 0$ has a 2-dimensional space of solutions and the size of the matrix $A$ is $6 \times 5$. What is the dimension of (a) the Null space of $A$? (b) the Column space of $A$? (c) the Row space of $A$? Justify your answers!

3. (15 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix}
-1 & -2 & -4 \\
0 & 0 & -1 \\
0 & 2 & 3
\end{pmatrix}$

is $-(\lambda - 1)(\lambda + 1)(\lambda - 2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that the matrix $A$ above satisfies

\[P^{-1}AP = D\]

4. (12 points) Determine for which of the following matrices $A$ below there exists an invertible matrix $P$ (with real entries) such that $P^{-1}AP$ is a diagonal matrix. You do not need to find $P$. Justify your answers!

(a) \[\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}\]

(b) \[\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}\]

(c) \[\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}\]

5. (20 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

Note: Parts 5a, 5b are mutually independent and are not needed for doing parts 5c, 5d, 5e.
(a) Find the distance between the two points \(v_1\) and \(v_2\) in \(\mathbb{R}^3\).
(b) Find a vector of length 1 which is orthogonal to \(W\).
(c) Find the projection of \(v_2\) to the line spanned by \(v_1\).
(d) Write \(v_2\) as the sum of a vector parallel to \(v_1\) and a vector orthogonal to \(v_1\).
(e) Find an orthogonal basis for \(W\).
(f) Find an orthogonal \(3 \times 3\) matrix \(U\), such that the corresponding linear transformation from \(\mathbb{R}^3\) to \(\mathbb{R}^3\) takes the \(x_1\) axis to the line spanned by \(v_1\) and the \(x_1, x_2\) coordinate plan to \(W\). Hint: Use parts 5b and 5d.

6. (16 points) Let \(W\) be the plane in \(\mathbb{R}^3\) spanned by \(u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\) and \(u_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\)

(a) Find the projection of \(b = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\) to \(W\).
(b) Find the distance from \(b\) to \(W\).
(c) Find a least square solution of the equation \(Ax = b\), where \(A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}\) is the \(3 \times 2\) matrix with columns \(u_1\) and \(u_1 + u_2\). I.e., find a vector \(x\) in \(\mathbb{R}^2\) which minimizes the length \(\|Ax - b\|\).
(d) Find the coefficients \(c_0, c_1\) of the line \(y(x) = c_0 + c_1x\) which best fits the three points \((x_1, y_1) = (1, 2), (x_2, y_2) = (-2, 1), (x_3, y_3) = (1, -2)\) in the \(x, y\) plane. The line should minimize the sum \(\sum_{i=1}^{3} [y(x_i) - y_i]^2\). **Justify your answer!**

7. (16 points) The vectors \(v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\) are eigenvectors of the matrix \(A = \begin{pmatrix} .4 & .6 \\ .6 & .4 \end{pmatrix}\).

(a) The eigenvalue of \(v_1\) is _____.

The eigenvalue of \(v_2\) is _____.

(b) Find the coordinates of \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) in the basis \(\{v_1, v_2\}\).

(c) Compute \(A^{50} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\).

(d) As \(n\) gets larger, the vector \(A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) approaches ____. **Justify your answer.**