Math 697: MIDTERM

Problem 1 (General random walk on \{0, \cdots, N\}) Let \(X_n\) be a Markov chain on the state space \{0, \cdots, N\} with a transition probabilities

\[
\begin{align*}
p(0,0) &= q_0, & p(0,1) &= p_0 \\
p(j,j-1) &= q_j, & p(j,j) &= r_j, & p(j,j+1) &= p_j, & j = 1, \cdots, N-1 \\
p(N,N-1) &= q_N, & p(N,N) &= p_N,
\end{align*}
\]

with \(p_0 + q_0 = p_N + q_N = 1\) and \(p_j + r_j + q_j = 1\) for \(j = 1, \cdots, N-1\) and we assume that \(p_j > 0\) and \(q_j > 0\) for all \(j\).

1. Show that the Markov chain \(X_n\) satisfies detailed balance, i.e., show that there exists positive number \(\nu(0), \cdots, \nu(N)\) such that

\[
\nu(i)p(i,j) = \nu(j)p(j,i).
\]

Use this to give a formula for the stationary distribution for \(x_n\) in terms of the \(p_j\)'s, \(q_j\)'s and \(r_j\)'s.

2. Consider the following Markov chain. An urn contains \(N\) balls which are either white or black. At each step one picks a ball in the urn at random and it is replaced with probability \(p\) by a white ball and with probability \(1-p\) by a black ball. Let \(X_n\) denotes the number of white balls after \(n\) steps. Compute the transition probabilities and the stationary distribution.

Problem 2 Let \(X_n\) be a positive recurrent Markov chain on the state space \(S\) with stationary distribution \(\pi\). Consider the stochastic process \(Y_n = (X_n, X_{n+1})\) with state space \(S \times S\).

1. Show that \(Y_n\) is a Markov chain.

2. Compute the transition probabilities and the stationary distribution of \(Y_n\).

3. Consider the Markov chain with state space \(1, 2, 3\) and transition probability

\[
P = \begin{pmatrix}
1/2 & 1/6 & 1/3 \\
1/2 & 1/4 & 1/4 \\
2/3 & 1/3 & 0
\end{pmatrix}
\]

Compute the long run proportion of steps for which \(X_{n+1} \geq X_n\).

Problem 3 (Partially observed Markov chains) Let \(X_n\) be an irreducible Markov chain with a finite state space \(S\) and transition matrix \(P = (p(i,j))\). Let \(T\) be a subset of states, \(T \subset S, T \neq S\). Let \(\nu_j, j \geq 0\), denote the successive times at which the Markov chain visits one of the states in \(T\), i.e.

\[
\begin{align*}
\nu_0 &= \inf \{ n \geq 0 : X_n \in T \}, \\
\nu_1 &= \inf \{ n > \nu_0 : X_n \in T \}, \\
\vdots
\end{align*}
\]
Define a new stochastic process $Y_j$ with state space $T$ which is given by

$$Y_j = X_{\nu_j}.$$ 

You can think of this process as follows: you can only observe $X_n$ only if $X_n$ is in one of the states in $T$. Moreover you don’t have a watch and thus have no way to keep track of the time elapsed between successive visits to $T$.

1. Show $Y_j$ is a Markov process.

2. Reordering the state if necessary we can assume that the transition matrix has the form as

$$P = \frac{T}{T^c} \begin{pmatrix} R & U \\ S & Q \end{pmatrix}$$

Let $D = (d_{ij})$ be the transition matrix for the Markov chain $Y_j$, i.e. $d_{ij} = P\{Y_1 = j \mid Y_0 = i\}$ for $i, j \in T$. Compute the matrix $D$ in terms of the matrix $R, U, S, Q$.

3. Suppose that the Markov chain $X_n$ has a stationary distribution $\pi = (\pi(1), \ldots, \pi(N))$. What is the stationary distribution for the Markov chain $Y_n$.

4. Compute

$$\lim_{k \to \infty} \frac{\nu_k}{k}.$$