NOTE 1: It is fine to work with other students in the class on this problem set, but when you hand it in, you must say with whom you worked. Each person’s write up must be done separately though. Identical answers (to relatively complicated problems) from different students will not be graded.

NOTE 2: If you use a source other than the textbook to do these problems, you must list the source. That said, it is never OK to copy from another source verbatim! If I notice that, I won’t grade the problem.

1. Reading: Please chapter 6 in Graybill. We will be here for a while. You may skip 6.8.

2. Let \( y = X\beta + \epsilon, \epsilon_i \sim N(0, \sigma^2) \), and suppose that \( X \) has full rank \((p + 1)\). Let \( C(X) = \{X\beta, \beta \in \mathbb{R}^{p+1}\} \). Let \( C(\text{perp})(X) = \{e, e^T v = 0 \text{ for any } v \in C(X)\} \).

(a) Show that \( I - X(X^T X)^{-1}X^T \) is a perpendicular projection matrix onto \( C(\text{perp})(X) \).

(b) Let \( \hat{y} = X\hat{\beta} = X(X^T X)^{-1}X^Ty \) and \( e = y - \hat{y} \).

i. Show that \( e^T \hat{y} = 0 \).

ii. Suppose that the first column of \( X \) is all 1s. Show that \( \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i \) and \( \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}_1 + \ldots + \hat{\beta}_p \bar{x}_p \).

iii. Prove that \( y^T y = \hat{y}^T \hat{y} + e^T e \).

iv. Prove that the previous result implies

\[
\sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n} (y_i - \hat{y}_i)^2.
\]

(Note that this is \( SST = SSR + SSE \).)

v. Prove that \( MSE = SSE/(n - p) \) is independent from \( \hat{\beta} \).
3. Let \( X \sim N(0, 1) \). Let \( Z \) be a random variable with \( Pr(Z = 0) = Pr(Z = 1) = 1/2 \), and let \( Y = (X + 1)^Z - (X + 1)^{1-Z} \).

(a) Show that the marginal distribution of \( Y \) is \( N(0, 1) \).

(b) Show that \( Y \) and \( X \) are not independent.

(c) Show that \( Y \) and \( X \) are uncorrelated.

(d) Is the joint distribution of \( X \) and \( Y \) multivariate normal? Why or why not?