Stat705 Exam 1

This exam is closed book and closed notes. All answers are worth an equal number of points.

1. Let $y_i = \mu + e_i, e_i \sim i.i.d. N(0, \sigma^2), i = 1, \ldots, n.$

(a) Write a matrix expression for $\hat{\mu},$ the least squares estimator of $\mu.$

(b) Consider $r_i = y_i - \hat{y}_i, i = 1, \ldots, n.$ (Note: $\hat{y}_i = \hat{\mu}$).

i. What is the distribution of $r = (r_1, \ldots, r_n)^T$?

ii. Are $r_1$ and $r_2$ independent? Why or why not?

iii. What is the expected value of the estimated variance of $r_i$? (i.e. $\hat{\text{var}}(r_i) = \frac{1}{n} \sum_{i=1}^{n} (r_i - \tau)^2$ where $\tau = \frac{1}{n} \sum_{i=1}^{n} r_i$) Please use a quadratic form to get the result. (Hint to simplify: What does $1^T r$ equal?)

2. Suppose $y \sim N(X\beta, \sigma^2 I), X$ is $n$ by $p$, rank $p < n$.

(a) Write down the pdf of $y$.

(b) Let $U$ be an $n$ by $n$ orthonormal matrix ($U^T U = UU^T = I_n$). Let $z = Uy$.

i. What is the distribution of $z$?

ii. Suppose $z$ is observed and $X$ and $U$ are known. What is the MLE of $\beta$?

3. Let $X = (1_n \ x \ x^2 \ x^3)$ and $\tilde{X} = (1_n \ x)$ where $x^k = (x_1^k, \ldots, x_n^k)^T$. Let $P = X(X^TX)^{-1}X^T$ and $\tilde{P} = \tilde{X}(\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T$. Assume the inverses exist. Let $F = (y^T(P - \tilde{P})y/2)/(y^T(I_n - P)y/(n - 4)).$

(a) What two models are compared by $F$?

(b) Suppose $F = 100,000$. Which model would you probably prefer and why?

(c) Let $y \sim N(\tilde{X}\beta, \sigma^2 I_n)$. Are the numerator and denominator of $F$ independent? Why or why not?