1. Consider the polynomial \( f(x) = x^2 - x - 2 \).

(a) Find \( P_1(x) \), \( P_2(x) \) and \( P_3(x) \) for \( f(x) \) about \( x_0 = 0 \). What is the relation between \( P_3(x) \) and \( f(x) \)? Why?

(b) Find \( P_1(x) \), \( P_2(x) \) and \( P_3(x) \) for \( f(x) \) about \( x_0 = 2 \). What is the relation between \( P_3(x) \) and \( f(x) \)? Why?

(c) In general, given a polynomial \( f(x) \) with degree \( \leq m \), what can you say about \( f(x) - P_n(x) \) for \( n \geq m \)?

**ANS:** First note that \( f'(x) = 2x - 1 \), \( f''(x) = 2 \), and \( f'''(x) \equiv 0 \). Then we have

(a) Let’s find \( P_3(x) \) which will also gives us \( P_1(x) \) and \( P_2(x) \). We have for \( x_0 = 0 \):

\[
P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3
= -2 + (-1)(x - 0) + \frac{2}{2}(x - 0)^2 + \frac{0}{6}(x - 0)^3
= -2 - x + x^2
\]

So, \( P_2(x) = -2 + (-1)(x - 0) + \frac{2}{2}(x - 0)^2 = -2 - x + x^2 \), and \( P_1(x) = -2 + (-1)(x - 0) = -2 - x \). \( P_3(x) = f(x) \) because \( f'''(x) \equiv 0 \), and thus we must have \( R_3(x) \equiv 0 \).

(b) Again, we find \( P_3(x) \) which also gives us \( P_1(x) \) and \( P_2(x) \). With \( x_0 = 2 \) we have:

\[
P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3
= 0 + 3(x - 2) + \frac{2}{2}(x - 2)^2 + \frac{0}{3!}(x - 0)^3
= -2 - x + x^2
\]

So, \( P_2(x) = 0 + 3(x - 2) + \frac{2}{2}(x - 2)^2 = -2 - x + x^2 \), and \( P_1(x) = 0 + 3(x - 2) = 3x - 6 \). And again, as in (a), \( P_3(x) = f(x) \) because \( f'''(x) \equiv 0 \).

(c) We will have that \( f(x) - P_n(x) \equiv 0 \) since \( f(x) \) is a polynomial of degree at most \( m \), thus \( f^{(n+1)}(x) \equiv 0 \) when \( n \geq m \), hence the error term is identically zero.
2. Find both $P_2(x)$ and $P_3(x)$ for $f(x) = \cos x$ about $x_0 = 0$, and use them to approximate $\cos(0.1)$. Show that in each case the remainder term provides an upper bound for the true error.

ANS: First note that $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f''''(x) = \cos x$. Let’s find $P_3(x)$ which will also gives us $P_2(x)$. We have for $x_0 = 0$:

$$P_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$= 1 + 0(x-0) + \frac{(-1)}{2}(x-0)^2 + \frac{0}{6}(x-0)^3$$

$$= 1 - \frac{x^2}{2}$$

Since $f^{(3)}(0) = 0$ we also have $P_2(x) = 1 - \frac{x^2}{2}$, so in this case $P_3(x) \equiv P_2(x)$. We have $P_3(0.1) = P_2(0.1) = 1 - (0.1)^2/2 = 1 - 1/200 = 199/200 = 0.995$. Since both Taylor polynomials are the same the error in both cases is

$$|\cos(0.1) - 0.995| \approx 4.165278025713981e-06.$$

From Taylor’s theorem, the error that results from using $P_2(x)$ as an approximate (note $n = 2$) is

$$|\cos 0.1 - 0.995| = \left| \frac{f^{(3)}(c_x)}{3!} (0.1 - 0)^3 \right| \text{ for } c_x \in (0, 0.1)$$

$$= |\sin(c_x)/6000| \text{ for } c_x \in (0, 0.1)$$

$$\leq \sin(0.1)/6000$$

$$\approx 1.663890277447136e-05$$

Now, again from Taylor’s Theorem, the error using $P_3(x)$ we have (note $n = 3$)

$$|\cos 0.1 - 0.995| = \left| \frac{f^{(4)}(c_x)}{4!} (0.1 - 0)^4 \right| \text{ for } c_x \in (0, 0.1)$$

$$= |\cos(c_x)/240000| \text{ for } c_x \in (0, 0.1)$$

$$\leq \cos(0)/240000$$

$$\approx 4.166666666666667e-06$$

So in each case an upper bound derived using the error term for Taylor polynomials is indeed larger than the actual error.
3. Find $P_2(x)$ for $f(x) = e^x \cos x$ expanded about $x_0 = 0$. Then find a bound on the error $|f(x) - P_2(x)|$ in using $P_2$ to approximate $f$ on $[0, 1]$.

**ANS:** First note that $f'(x) = e^x(\cos x - \sin x)$, $f''(x) = -2e^x \sin x$, and $f'''(x) = 4e^x \cos x + 2e^x \sin x$.

We have for $x_0 = 0$:

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 1 + 1(x - 0) + \frac{0}{2}(x - 0)^2 = 1 + x$$

To bound the error in using $P_2(x)$ to approximate $f(x)$ over the interval $[0, 1]$, from Taylor’s theorem we have

$$|f(x) - P_2(x)| = \left| f^{(3)}(c) \frac{(x - 0)^3}{3!} \right| \text{ for } c \in (0, 1)$$

$$= \left| -2e^{c_2}(\cos (c_2) + \sin (c_2)) (x - 0)^3 \right| \text{ for } c_2 \in (0, 1)$$

$$\leq \left| 2e^{c_2}(\cos (c_2) + \sin (c_2)) \right| \frac{(x - 0)^3}{3!} \text{ for } c_2 \in (0, 1)$$

since $x^3$ is maximized at $x = 1$. We are left to find an upper bound for the expression in the numerator, i.e., we have to find the maximum absolute value of $g(x) = 2e^x(\cos x + \sin x)$ over the interval $[0, 1]$. From Calculus we know this can only occur at the endpoints of the interval, $x = 0, 1$, or in $(0, 1)$ where $g'(x) = 4e^x \cos x = 0$. But $4e^x > 0$ and there is no point in $[0, 1]$ where $\cos x = 0$ (check this, if need be, by drawing a picture!).

So checking $x = 0, 1$ (the endpoints of the interval) we have

$$|g(0)| = |2e^0(\cos (0) + \sin (0))| = 2,$$

and

$$|g(1)| = |2e^1(\cos (1) + \sin (1))| \approx 7.512098454189456.$$ 

Using the larger of these one can conclude that

$$\max_{x \in [0,1]} |f(x) - P_2(x)| \leq \frac{7.512098454189456}{6} \approx 1.252016409031576.$$
4. Consider \( f(x) = e^x \), and find a general formula for the Taylor polynomial \( P_n(x) \) for \( f \) about \( x_0 = 0 \).

(a) Using the remainder term, find a minimum value of \( n \) necessary for \( P_n(x) \) to approximate \( f(x) \) to within \( 10^{-6} \) on \([0,0.5]\).

(b) Is \( f(x) \) analytic on \((-\infty, \infty) = \mathbb{R}\)? Prove your answer.

**ANS:** Note that \( f^{(n)}(x) = e^x \) so for \( n \geq 0 \), with \( x_0 = 0 \), we have

\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k
\]

\[
= \sum_{k=0}^{n} \frac{1}{k!} x^k
\]

\[
= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}
\]

(a) The remainder term is given by \( R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1} = \frac{e^{c_x}}{(n+1)!} x^{n+1} \) for \( c_x \in (0,0.5) \), so we need to find the minimum value of \( n \) such that

\[
\max_{x \in [0,0.5]} |R_n(x)| = \max_{x \in [0,0.5]} \frac{e^{c_x}}{(n+1)!} x^{n+1} \leq \frac{e^{1/2}}{(n+1)!} \frac{1}{2^{n+1}} \leq 10^{-6},
\]

or we need the minimum \( n \) such that

\[
2^{n+1} (n+1)! \geq e^{1/2} \times 10^6 \approx 1648721.270700128
\]

Just trying some values of \( n \) on the right one sees that \( 2^7 \times 7! = 654120 \) and \( 2^8 \times 8! = 10321920 \), so with \( n + 1 = 8 \) we see that one must have \( n \geq 7 \).

(b) We need to show that for each value of \( x \in \mathbb{R} \) that

\[
\lim_{n \to \infty} |e^x - P_n(x)| = \lim_{n \to \infty} |f(x) - P_n(x)| = \lim_{n \to \infty} |R_n(x)| = \lim_{n \to \infty} \left| \frac{f^{n+1}(c_x)}{(n+1)!} (x-0)^{n+1} \right| = 0,
\]

To do so, **FIX** an \( x \in \mathbb{R} \) and note that there must exist a **positive** integer \( M \) (i.e., \( M \in \{1,2,3,4,\ldots\} \)) such that \( M > |x| \). Why? Because \( x \) is fixed. Suppose \( x = \pm 134,665,323,33452 \), take \( M = 200,000,000 \) if you like, or \( M = 134,665,324 \). Also, since \( c_x \) lies in the interval between \( x \) and \( x_0 = 0 \), then \(|c_x| < M\). So once \( n > M \),

\[
|R_n(x)| = \left| \frac{e^{c_x}}{(n+1)!} (x-0)^{n+1} \right| \leq \left| \frac{e^M}{(n+1)!} M^{n+1} \right| = e^M \times \frac{M \times M \times M \times \cdots \times M}{1 \times 2 \times 3 \times \cdots \times (n+1)}
\]

\[
= e^M \times \left( \frac{M}{1} \times \frac{M}{2} \times \frac{M}{3} \times \cdots \times \frac{M}{M-1} \times \frac{M}{M} \times \frac{M}{M+1} \times \frac{M}{M+2} \times \cdots \times \frac{M}{(n+1)} \right)
\]

\[
= e^M \times \left( \frac{M}{1} \times \frac{M}{2} \times \frac{M}{3} \times \cdots \times \frac{M}{M-1} \times \frac{M}{M} \right) \times \left( \frac{M}{M+1} \times \frac{M}{M+2} \times \cdots \times \frac{M}{(n+1)} \right)
\]

Note that since \( x \) is fixed the first two terms above are **bounded**. Finally, each ratio in the last term is less than 1 and as \( n \to \infty \) this term goes to 0, completing the proof.
5. Consider \( f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \).

(a) Show that \( f(x) \in C^\infty(\mathbb{R}) \).

(b) Find a general formula for the Taylor polynomial \( P_n(x) \) for \( f \) about \( x_0 = 0 \).

(c) Is \( f(x) \) analytic on \( (-\infty, \infty) = \mathbb{R} \)? Prove your answer.

**ANS:** (a) For \( x \neq 0 \), \( f(x) \) is the composition of two \( C^\infty \) function and thus is \( C^\infty \) itself. For \( x = 0 \), note that

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-x^{-2}} = e^{-\lim_{x \to 0} x^{-2}} = 0,
\]

hence \( f(x) \) is continuous at 0. Again for \( x \neq 0 \), \( f'(x) = 2x^{-3}e^{-x^{-2}} \), and induction shows that \( f^{(n)}(x) = x^{-3n}P(x)e^{-x^{-2}} \) where \( P(x) \) is a polynomial of degree \( 2(n-1) \). This formula shows that \( \lim_{x \to 0} f^{(n)}(x) = 0 \), thus taking \( f^{(n)}(0) = 0 \) shows that \( f(x) \in C^\infty(\mathbb{R}) \).

(b) Since \( f^{(n)}(0) = 0 \) for \( n = 0, 1, \ldots \), it is easy to see \( P_n(x) = 0 \) for every \( n \).

(c) From (b) it is clear that the Taylor series of \( f(x) \) about \( x_0 = 0 \) does not equal the function for any \( x \neq 0 \). Since we have found a \( x_0 \in \mathbb{R} \) at which \( f(x) \) does not equal its Taylor series about \( x_0 \), \( f(x) \) is **not** analytic.