Lecture #6

HW #1 due Wednesday

Read 3.1-3.4. Chapter 3 pdf is available on the course web page.

Algorithm unstable → Goal: stable
ill-conditioned → well-conditioned

\[ F(x) = \sqrt{x + 1} - \sqrt{x} \]
\[ F(1984) = \sqrt{1985} - \sqrt{1984} \]
\[ F(x) = \sqrt{x + 1} - \sqrt{x} \left( \frac{\sqrt{x+1} + \sqrt{x}}{2x+1} \right) \]
\[ = \frac{1}{\sqrt{x+1} + \sqrt{x}} \]

Roots of Nonlinear Equations

Defn: Given a function \( F(x) \), \( a \) is called a root (or zero) of \( F(x) \) if \( F(a) = 0 \).

Ex: \( F(x) = x^2 - 3x + 2 = (x-1)(x-2) = 0 \Rightarrow \) roots are \( a_1 = 1 \), \( a_2 = 2 \).

Ex: \( F(x) = 2 - e^x = 0 \Rightarrow x = \ln(2) = 0.6931471806 \).

Question: Given a general nonlinear function, how do we find the roots?

\[ Ax = b \rightarrow Ax - b = 0 \]
\[ \max \]
\[ P_N(x) = \det(A - xI) \]
characteristic eqn
Then [(Intermediate Value Theorem) (IVT)]

Suppose \( F(x) \) is continuous on \([a, b]\). Let \( k \) be any number between \( F(a) \) and \( F(b) \). Then there exists an \( x \in (a, b) \) such that \( F(x) = k \).

Application to root finding: If we have \( a \) and \( b \) such that

\[
\begin{align*}
F(a) &< 0 \quad \text{and} \quad F(b) > 0 \\
\text{or} \quad F(a) > 0 \quad \text{and} \quad F(b) < 0
\end{align*}
\]

⇒ by the IVT, there is a root \( x \in (a, b) \).

Bisection method:

\( F(a) \cdot F(b) < 0 \)

Idea: Check the sign of \( F(\frac{a+b}{2}) \) and shrink the interval!

Algorithm (assumes \( F(a) \cdot F(b) < 0 \))

\[
\begin{align*}
a_0 &= a, \quad b_0 = b \\
n &= 0 \\
x_n &= (a_n + b_n)/2 \\
\text{If} & \quad F(x_n) \cdot F(a_n) < 0, \text{then} \\
\quad & \quad a_{n+1} = a_n \\
\quad & \quad b_{n+1} = x_n \\
\text{Else} & \quad a_{n+1} = x_n \\
\quad & \quad b_{n+1} = b_n \\
\text{end} & \\
n &= n+1
\end{align*}
\]

**Example:**

\( F(x) = x^2 - 3 \), \( F(1) = -2 \) is a root

\( F(2) = 1 \) in \([1, 2]\), i.e.

\( x = \sqrt{3} \approx 1.73205 \)

| \( n \) | \( a_n \) | \( b_n \) | \( x_n \) | \( F(x_n) \) | \( |x_n - x| \) |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 1.5 | -0.75 | 0.2321 |
| 1 | 1.5 | 2 | 1.75 | 0.0625 | 0.0179 |
| 2 | 1.5 | 1.75 | 1.625 | 0.3594 | 0.0171 |
| 3 | 1.625 | 1.75 | 1.6875 | 0.01523 | 0.00446 |
| 4 | 1.6875 | 1.75 | 1.71875 | 0.00459 | 0.00133 |

\( \alpha \rightarrow \alpha \rightarrow 0 \)}
Question: When should we stop? 

(1) \(|a-x_0|\) is small \(\Rightarrow\) we do not have access to this info.

(2) \(\|F(x_n)\|\) is small \(\Rightarrow\) 

\[ x_0, x_1, x_2, \ldots \rightarrow x_0 \] 

Def. (Convergence of a sequence)

A sequence \(x_n\) converges to \(x\) if given \(\varepsilon > 0\) there exists an \(N\) such that \(|x_n - x| < \varepsilon\) when \(n \geq N\).

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Theorem: Error estimates for bisection

Suppose \(F(x)\) is continuous on \([a,b]\) and \(F(a) \cdot F(b) < 0\), thus by the IVT there is an \(x \in (a,b)\) such that \(F(x) = 0\). If \(x_{n+1}\) is generated by bisection, then

\[ |x - x_n| \leq \frac{b-a}{2^n} \text{ for all } n \geq 0 \]

Proof:

\[ a_n \quad x_{n+1} \quad b_n \]

\[ |x - x_n| \leq |b_n - a_n| = \frac{1}{2^n} |b_{n-1} - a_{n-1}| \]

\[ = \frac{1}{2^n} |b_{n-2} - a_{n-2}| \]

\[ = \frac{1}{2^n} |a_0 - b_0| = \frac{|a-b|}{2^n} \]