HW#3 Solutions

April 2, 2014

1. Suppose $O \subseteq B_r(x)$ for some ball $B_r(x)$ with radius $r$ and centre $x$. Then $O$ is also contained in the ball, $B_{r+|x|}(0)$, of radius $r + |x|$ and centre 0.

2. By Heine-Borel, if the set $A \in \mathbb{R}^n$ is not compact then it is either not bounded or not closed (or both of course!).

Suppose $A$ is not bounded. The function $f(x) = |x|$ is continuous on the whole of $\mathbb{R}^n$, thus it is also continuous on $A$. But it is clearly not bounded.

Suppose $A$ is not closed. Let $a \in \bar{A} - A$, (why does such an $a$ exist?). The function $g(x) = \frac{1}{|x-a|}$ is well defined on $A$. Now, since $a \in \bar{A} - A$ we have that

$$A \cap B_r(a) \neq \emptyset, \text{ for all } r > 0,$$

i.e. for every $r > 0$, we can find $x \in A$ such that $|x - a| < r$. Hence, $g$ is not bounded.

6a. Let $f(x) = |x|e^{-|x|}$, thus

$$f(x) = \begin{cases} x e^{-x} & \text{if } x > 0 \\ -xe^x & \text{if } x < 0 \end{cases}$$

differentiating gives,

$$f'(x) = \begin{cases} e^{-x}(1-x) & \text{if } x > 0 \\ e^x(-1-x) & \text{if } x < 0 \end{cases}.$$ 

Note, $f$ is differentiable everywhere except for $x = 0$. Moreover, $f'(x) = 0$ iff $x = 1, 0$ or $-1$. Note, $f''(1) = (1-2)e^{-1} < 0$, thus $x = 1$ is a local maximum. Since $x = 1$ is the only critical point on $\{x > 0\}$ and $f$ is continuous on $\{x > 0\}$, we must have that $x = 1$ is a global maximum on $\{x > 0\}$.

b. The maximum points are $(\pm 1, \frac{1}{2}).$

c. Since $(\pm 1, \frac{1}{2})$ are global maximums, we must have that $\text{Im}(f) \subseteq (-\infty, \frac{1}{2}]$. But $f(x) \geq 0$ for all $x$, moreover, $f(0) = 0$. Thus $\text{Im}(f) = [0, \frac{1}{2}]$. 

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7. Consider the function \( g(x) = f(x) - mx \). This is continuous on \([a, b]\), which is compact, thus \( g \) attains a minimal value at some point \( c \in [a, b] \).

We now need to show \( c \neq a \) and \( c \neq b \). Note,

\[
g'(a) = f'(a) - m < 0.
\]

Thus,

\[
\lim_{h \to 0} \frac{g(a + h) - g(a)}{h} < 0.
\]

This implies that for some \( \tilde{h} > 0 \), which is small enough we have

\[
\frac{g(a + \tilde{h}) - g(a)}{\tilde{h}} < 0,
\]

thus, \( g(a) \) is not minimum.

Similarly,

\[
g'(b) = f'(b) - m > 0
\]

thus there exists some \( \tilde{h} < 0 \), which is small enough we have

\[
\frac{g(b + \tilde{h}) - g(b)}{\tilde{h}} > 0,
\]

Thus \( g(b + \tilde{h}) < g(b) \).

So we have shown that \( c \in (a, b) \). Since \( c \) is a min point, we must have that

\[
g'(c) = f'(c) - m = 0
\]

i.e. \( f'(c) = m \). As required.

11. Let \( p(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0 \) be a real polynomial with \( k \) odd. We define \( C, A \in \mathbb{R} \) as

\[
C = \max\{1, |a_{k-1}|, \ldots, |a_0|\},
\]

\[
A = kC + 1.
\]

Now we note that if \( x \leq -A \), then

\[
p(x) \leq (-A)^k + CA^{k-1} + \ldots + CA + C
\]

\[
\leq -A^k + kCA^{k-1}
\]

\[
= A^{k-1}(-A + kC) = -A^{k-1} < 0.
\]

Similarly, if \( x \geq A \)

\[
p(x) \geq A^k - CA^{k-1} - \ldots - CA - C
\]

\[
\geq A^k - kCA^{k-1}
\]

\[
= A^{k-1}(A - kC) = A^{k-1} > 0.
\]

Thus, we have shown that \( p(-A) \leq 0 \) and \( p(A) \geq 0 \). By IVT, there exists \( y \in [-A, A] \) such that \( p(y) = 0 \).