1. (1.5.2) For each of the following subsets, state whether it is open or closed (or both or neither), and say why.
   a. \((x, y)\)-plane in \(\mathbb{R}^3\) \textbf{ans:} closed
   b. \(\mathbb{R} \subseteq \mathbb{C}\) \textbf{ans:} closed
   c. the line \(x = 5\) in the \((x, y)\)-plane \textbf{ans:} closed
   d. \((0, 1) \subseteq \mathbb{C}\) \textbf{ans:} neither
   e. \(\mathbb{R}^n \subseteq \mathbb{R}^n\) \textbf{ans:} open and closed
   f. the unit sphere in \(\mathbb{R}^3\) \textbf{ans:} closed (note: the unit sphere is the set \(x^2 + y^2 + z^2 = 1\))

2. (1.5.4) a. Show that the interior of \(A\) is the biggest open set contained in \(A\).
   b. Show that the closure of \(A\) is the smallest closed set that contains \(A\).
   c. Show that the closure of a set \(A\) is \(A\) plus its boundary: \(\overline{A} = A \cup \partial A\).
   d. Show that the boundary is the closure minus the interior: \(\partial A = \overline{A} - \mathring{A}\).
   \textbf{ANS:}

3. (1.5.7) For each of the following formulas, find its natural domain, and show whether it is open, closed or neither.
   \textbf{ANS:}

4. (1.5.8) a. Find the inverse of the matrix \(B = \begin{bmatrix} 1 & \epsilon & \epsilon \\ 0 & 1 & \epsilon \\ 0 & 0 & 1 \end{bmatrix}\) by finding the matrix \(A\) such that \(B = I - A\) and computing the value of the series \(S = I + A + A^2 + A^3 + \cdots\).
   b. Compute the inverse of the matrix \(C = \begin{bmatrix} 1 & -\epsilon \\ +\epsilon & 1 \end{bmatrix}\) where \(|\epsilon| < 1\).
   \textbf{ANS:} a. \(A = I - B = \begin{bmatrix} 0 & -\epsilon & -\epsilon \\ 0 & 0 & -\epsilon \\ 0 & 0 & 0 \end{bmatrix}\). Then \(A^2 = \begin{bmatrix} 0 & 0 & \epsilon^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\) and \(A^n = 0\) for \(n \geq 3\),
   so \(S = I + A + A^2 = \begin{bmatrix} 1 & -\epsilon & -\epsilon + \epsilon^2 \\ 0 & 1 & -\epsilon \\ 0 & 0 & 1 \end{bmatrix}\), and it is easy to check that \(SB = I\).
   b. First, we can use our formula for the inverse of a \(2 \times 2\): \(C^{-1} = \frac{1}{1+\epsilon^2} \begin{bmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{bmatrix}\), which we can use to check our answer that we get below.
Writing $C = I - A$ gives $A = \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix}$, and we have $|A| < 1$ since $|\epsilon| < 1$. Then

$$C^{-1} = I + A + A^2 + A^3 + A^4 + A^5 + ...$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} + \begin{bmatrix} -\epsilon^2 & 0 \\ 0 & -\epsilon^2 \end{bmatrix} + \begin{bmatrix} 0 & -\epsilon^3 \\ \epsilon^3 & 0 \end{bmatrix} + \begin{bmatrix} \epsilon^4 & 0 \\ 0 & \epsilon^4 \end{bmatrix} + \begin{bmatrix} 0 & \epsilon^5 \\ -\epsilon^5 & 0 \end{bmatrix} + ...$$

$$= \begin{bmatrix} 1 - \epsilon^2 + \epsilon^4 - \epsilon^6 + \epsilon^8 ... & \epsilon - \epsilon^3 + \epsilon^5 - \epsilon^7 + \epsilon^9 - ... \\ -\epsilon + \epsilon^3 - \epsilon^5 + \epsilon^7 - \epsilon^9 - ... & 1 - \epsilon^2 + \epsilon^4 - \epsilon^6 + \epsilon^8 + ... \end{bmatrix}$$

Now, recall that if $|x| < 1$ then we have the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$, so letting $x = -\epsilon^2$ we see that

$$\frac{1}{1 + \epsilon^2} = 1 - \epsilon^2 + \epsilon^4 - \epsilon^6 + \epsilon^8 - ... = 1 + x + x^2 + x^3 + x^4 + ... = \frac{1}{1 - x}$$

And factoring out an $\epsilon$ from the $(1,2)$ and $(2,1)$ entries shows gives $\frac{\epsilon}{1 + \epsilon^2}$ as the sum.

Thus, $C^{-1} = \frac{1}{1 + \epsilon^2} \begin{bmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{bmatrix}$.

5. **(1.5.13)** Prove the converse of proposition 1.5.17 (i.e., prove that if every convergent sequence in a set $C \subset \mathbb{R}^n$ converges to a point in $C$, the $C$ is closed.)

**ANS:** *(from the soln manual and class)* We will show that $\mathbb{R}^n - C$ is open, hence $C$ is closed.

Choose a point $a \in \mathbb{R}^n - C$, and **suppose** that

$$B_{1/n}(a) \cap C \neq \emptyset$$

for every positive integer $n$.

For each $n = 1, 2, ...$ choose $a_n \in B_{1/n}(a) \cap C$. Then the sequence $(a_n)_{n=1}^{\infty}$ converges to $a$, since for any $\epsilon > 0$ we can find $N$ such that for $n > N$ we have $1/n < \epsilon$, so for $n > N$ we have $|a - a_n| < 1/n < \epsilon$. Then our hypothesis implies that $a \in C$, a **contradiction**. Thus there must exist at least one $N$ such that $B_{1/N}(a) \cap C = \emptyset$. So we found a open neighborhood around $a$ that does not intersect $C$, hence $\mathbb{R}^n - C$ is open and thus $C$ is closed.

6. **(1.5.14)** State whether the following limits exist, and prove it.

a. $\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \frac{x^2}{x + y}$ **ans:** as there are no division by 0 issues and both the numerator and denominator are continuous at the point, so we simply evaluate the function at the point to get $\frac{1}{3}$. 

b. \( \lim \frac{\sqrt{|x|y}}{x^2 + y^2} \) \textbf{ans:} Let’s approach the origin along the line \( y = mx \) for \( m \neq 0 \). We then have

\[
\frac{\sqrt{|x|y}}{x^2 + y^2} = \frac{\sqrt{|x|mx}}{x^2 + m^2x^2} = \frac{mx\sqrt{|x|}}{x^2(1 + m^2)} = \frac{m\sqrt{|x|}}{x(1 + m^2)}
\]

so as \((x, y)^T \rightarrow (0, 0)^T\) by letting \( x \rightarrow 0 \), we see that the quantity goes to \( \pm\infty \) since \( 1/\sqrt{x} \) goes to \( \infty \) as \( x \rightarrow 0 \), thus the limit \textbf{DNE}.

c. \( \lim \frac{\sqrt{|xy|}}{x^2 + y^2} \) \textbf{ans:} Let’s approach the origin along the line \( y = mx \) for \( m \neq 0 \). We then have

\[
\frac{\sqrt{|xy|}}{x^2 + y^2} = \frac{\sqrt{|xmx|}}{\sqrt{x^2 + m^2x^2}} = \frac{|x|\sqrt{|m|}}{|x|\sqrt{1 + m^2}} = \frac{\sqrt{|m|}}{\sqrt{1 + m^2}}
\]

so as \((x, y)^T \rightarrow (0, 0)^T\) by letting \( x \rightarrow 0 \), we see that the answer depends on the slope \( m \), thus the limit \textbf{DNE}.

d. \( \lim \frac{x^2 + y^3 - 3}{x + y} \) \textbf{ans:} As in part a above, there are no division by 0 issues and \( f(x, y) \) is a polynomial which is continuous at every point, so we simply evaluate the function at the point to get \( 1^2 + 2^3 - 3 = 6 \).

7. \textbf{(1.5.18)} Prove proposition 1.5.19: If a sequence \( \{a_k\}_{k=1}^{\infty} \) converges to \( a \), then any subsequence converges to the same limit.

\textbf{ANS:} Let \( \{a_{i(k)}\}_{k=1}^{\infty} \) be a subsequence of the given sequence, and suppose the \( \lim_{k \rightarrow \infty} a_{i(k)} \neq a \).