(1) Let $A$ be an $n \times n$ matrix and $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = Ax$ the linear transformation with matrix $A$. What does it mean to say that a vector $v \in \mathbb{R}^n$ is an eigenvector of $A$ (or $T$) with eigenvalue $\lambda$?

(2) Arguing geometrically, describe the eigenvalues and eigenvectors of the following linear transformations.

(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = 2x$.

(b) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $y = 3x$.

(c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ the horizontal shear given by $T(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$.

(3) Let $A$ be an $n \times n$ matrix. Here is the strategy to find the eigenvalues and eigenvectors of $A$:

(a) Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

(b) For each eigenvalue $\lambda$ solve the equation $(A - \lambda I)v = 0$ to find the eigenvectors $v$ with eigenvalue $\lambda$.

[Why does this work? The equation $(A - \lambda I)v = 0$ is obtained from the equation $Av = \lambda v$ by rearranging the terms. This equation has a nonzero solution $v \in \mathbb{R}^n$ exactly when $(A - \lambda I)$ is not invertible, equivalently $\det(A - \lambda I) = 0$.]

1
The function $\det(A - \lambda I)$ is a polynomial of degree $n$ in the variable $\lambda$. In particular if $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$
\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}
$$

$$
= (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)
$$

and we can solve the characteristic equation using the quadratic formula. If $n = 3$ we can determine the polynomial $\det(A - \lambda I)$ by computing the determinant using either Sarrus’ rule or expansion along a row or column.

(4) For each of the following matrices, find all the eigenvalues and eigenvectors.

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(5) Let

$$
A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}
$$
The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x) = Ax$ is given by rotation about a line $L$ through some angle $\theta$. Find the line $L$.

[Hint: A vector $v$ in the direction of $L$ is an eigenvector of $A$ (why?). What is the corresponding eigenvalue?]

(6) Let $A$ be an $n \times n$ matrix. We say $A$ is diagonalizable if there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A$. In this case, let $\mathcal{B} = (v_1, \ldots, v_n)$ be the basis of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the $\mathcal{B}$-matrix of the transformation $T(x) = Ax$ is the diagonal matrix $D$ with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$ (why?). Equivalently, writing $S$ for the matrix with columns the vectors $v_1, \ldots, v_n$, we have

$$A = SDS^{-1}.$$ 

We can determine whether $A$ is diagonalizable as follows: for each eigenvalue $\lambda$, find a basis of the eigenspace $E_\lambda = \ker(A - \lambda I)$ (the subspace of $\mathbb{R}^n$ consisting of all the eigenvectors with eigenvalue $\lambda$ together with the zero vector). Now combine the bases of all the eigenspaces. These vectors are linearly independent. If there are $n$ vectors, then they form a basis $\mathcal{B}$ of $\mathbb{R}^n$ and $A$ is diagonalizable, otherwise $A$ is not diagonalizable.

(7) For each of the matrices $A$ of Q4, determine whether $A$ is diagonalizable. If $A$ is diagonalizable identify a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A$ and write down the $\mathcal{B}$-matrix of the linear transformation $T(x) = Ax$.

(8) For which values of $a$ and $b$ is the matrix $A = \begin{pmatrix} 2 & a \\ 0 & b \end{pmatrix}$ diagonalizable?

(9) If $A$ is diagonalizable we can compute an explicit formula for powers of $A$ as follows: Write $A = SDS^{-1}$ as above where $D$ is the diagonal matrix with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any positive integer $k$ we have

$$A^k = SD^k S^{-1}$$

(why?) and $D^k$ is the diagonal matrix with diagonal entries $\lambda_1^k, \ldots, \lambda_n^k$.

(10) For the matrices $A$ of Q4(a) and (b) compute a formula for $A^k$.
(11) Let $W \subset \mathbb{R}^3$ be the subspace with basis $B = (v_1, v_2)$ where
\[
v_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.
\]

(a) Using the Gram-Schmidt process, find an orthonormal basis $C = (u_1, u_2)$ for $W$.

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by orthogonal projection onto $W$. Write down a formula for $T(x)$ in terms of $u_1$ and $u_2$, and use it to compute $T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(12) Let $W \subset \mathbb{R}^4$ be the subspace with basis $B = (v_1, v_2)$ where
\[
v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \\ 1 \end{pmatrix}.
\]

(a) Using the Gram-Schmidt process, find an orthonormal basis $C = (u_1, u_2)$ of $W$.

(b) Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be orthogonal projection onto $W$. Compute $T \begin{pmatrix} 3 \\ 5 \\ 1 \\ 3 \end{pmatrix}$.

(13) Let
\[
u_1 = \frac{1}{9} \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix}, \quad u_2 = \frac{1}{9} \begin{pmatrix} -7 \\ 4 \\ -4 \end{pmatrix}, \quad u_3 = \frac{1}{9} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}.
\]

(a) Show that $B = (u_1, u_2, u_3)$ is an orthonormal basis of $\mathbb{R}^3$.

(b) Let $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Using part (a) or otherwise, compute the $B$-coordinate vector $[v]_B$ of $v$. 

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(14) Find all solutions of the following system of linear equations. Write your answer as a linear combination of vectors in $\mathbb{R}^5$.

\begin{align*}
  x_1 - x_2 + x_3 + 2x_5 &= 1 \\
  2x_1 - x_2 + 4x_3 + x_4 + 3x_5 &= 3 \\
  -x_1 + 3x_2 + 3x_3 + 5x_4 - x_5 &= 7
\end{align*}

(15) Let $V$ be a linear space and $T: V \to V$ a function (or transformation) from $V$ to $V$. What does it mean to say that $T$ is linear? (There are two conditions that must be satisfied.) If $T$ is linear what is $T(0)$?

(16) What does it mean to say that a subset $W \subset \mathbb{R}^n$ is a subspace? (There are 3 conditions that must be satisfied.) If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\lambda \in \mathbb{R}$, let $W$ be the subset of $\mathbb{R}^n$ consisting of all the vectors $v$ such that $T(v) = \lambda v$. Show that $W$ is a subspace of $\mathbb{R}^n$. [Remark: The subspace $W$ is the eigen-space $E_\lambda$ consisting of all the eigenvectors of $T$ with eigenvalue $\lambda$ together with the zero vector.]

(17) What is the rank-nullity theorem? If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, what can you say about the kernel of $T$ if $n > m$?

(18) Let $V$ be a linear space and $\mathcal{B}$ a basis of $V$. Let $T: V \to V$ be a linear transformation. What is the $\mathcal{B}$-matrix of $T$ and how can it be computed? In each of the following examples, write down a basis $\mathcal{B}$ of $V$, compute the $\mathcal{B}$-matrix of $T$, and determine whether $T$ is an isomorphism.

(a) $V = \mathcal{P}_2$, the linear space of polynomials $f(x)$ of degree $\leq 2$, and $T: V \to V, T(f(x)) = f(x) + f'(x) + f''(x)$.

(b) $V = \mathbb{R}^{2\times2}$, the linear space of $2\times2$ matrices, and $T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$, $T(X) = AX + XB$ where $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. 
