Problem 1. Consider the hyperelliptic Riemann surface $M$ whose affine part is defined by

$$w^2 = P(z)$$

with $P(z)$ a polynomial of odd degree $2d - 1$ and distinct zeros. We know from previous problems that

$$\pi : M \to \mathbb{CP}^1 : (z, w) \mapsto z$$

(with 2 points over infinity added) is a branched degree 2 cover (branched at the zeros of $P(z)$ with branch number 2). Only for $d \leq 2$ was $M$ a non-singular curve in $\mathbb{CP}^2$. Show that $M \subset O(d)$, for any $d \geq 1$, can be regarded as a non-singular curve (smooth 1-dim complex submanifold) in the (complex 2 dimensional) total space of the line bundle $O(d) \to \mathbb{CP}^1$. The map $\pi : M \to \mathbb{CP}^1$ then is the restriction of the bundle projection.

Hints: first, a submanifold $M \subset N$ of codimension $k$ (or dimension $\dim N - k$) of an ambient manifold $N$ is a subset which can be covered by open subsets $U \subset N$ (of the ambient manifold) with smooth (or holomorphic) maps $g_U : U \to \mathbb{R}^k$ whose zeros set $\{g_U = 0\} = U \cap M$ and whose differentials $dg_U$ have $\mathbb{F}$-rank $k$ at all points on $U \cap M$. One says $M$ is locally described by $g_U$. Special examples we already know about are $N = \mathbb{CP}^2$ and $M$ the zero set of a homogeneous polynomial $g$ under the right conditions on $g$ – see previous HW sheet. Generally a submanifold $M \subset N$ need not be described by one function only, and one needs different local functions for different parts of $M$. Second, in the equation $y^2 = P(z)$ look what happens at infinity, i.e., $z \mapsto 1/z$, and relate this to the transition function of $O(d)$...

Problem 2. Consider again the hyperelliptic Riemann surface $M$ defined by

$$w^2 = P(z)$$

with $P(z)$ a polynomial of odd degree $2d - 1$ and distinct zeros. Determine all holomorphic differentials on $M$, i.e. write down a basis for $H^0(K)$ where $K$ is the canonical bundle of $M$.

Hint: $z, w$ are holomorphic functions and $dz$ is a holomorphic differential on the affine part of $M$....check what happens at the points over infinity.

Problem 3. Show that there are no non-zero holomorphic differentials on $\mathbb{CP}^1$. 
Recall the notion of a point bundle $L(q) \to M$ on a Riemann surface $M$: if $q \in M$ let $q \in U \subset M$ be an open disk (i.e. $U = z^{-1}(D)$ of a disk in $\mathbb{C}$ under a centered chart $z : U \to \mathbb{C}$) and consider the open cover $U_1 := U$, $U_2 := M \setminus \{q\}$ of $M$. Then $U_1 \cap U_2 = U \setminus \{p\}$ and $L(q) \to M$ is the holomorphic line bundle given by the transition function $g_{12} : U \setminus \{p\} \to \mathbb{C}^\times$, $g_{12}(p) := z(p)$.

**Problem 4.** Show:

(i) The (holomorphic) isomorphism class of $L(q)$ does not depend on the choice of chart $(U, z)$ or the choice of disk $D$.

(ii) All point bundles on $\mathbb{CP}^1$ are (holomorphically) isomorphic.

(iii) Two point bundles $L(q_1)$ and $L(q_2)$ on a torus $C/\Gamma$ are (holomorphically) isomorphic if and only if $q_1 = q_2$.

The next problems go some way in showing that holomorphic line bundles over open disks in $\mathbb{C}$ are holomorphically trivializable. This proof is based on the solution of the delbar problem on disks. First some notation (which Tetsuya already introduced): the complexified cotangent bundle of a Riemann surface $TM^* \otimes_{\mathbb{R}} \mathbb{C} = \text{Hom}_\mathbb{R}(TM, \mathbb{C}) = K \oplus \bar{K}$: $\alpha = \alpha^{1,0} + \alpha^{0,1}$ splits into the canonical and anti-canonical complex line bundle: $\alpha \in K$ if and only if $\alpha \circ J = i\alpha$ etc. (this is nothing other than the decomposition of an $\mathbb{R}$-linear map $\alpha : T_pM \to \mathbb{C}$ into complex linear and complex antilinear parts). Correspondingly, the $\mathbb{C}$-valued smooth 1-forms split

$$\Omega^1(M, \mathbb{C}) = \Gamma(K) \oplus \Gamma(\bar{K}) = \Omega^{1,0}(M, \mathbb{C}) \oplus \Omega^{0,1}(M, \mathbb{C})$$

Using those notations, we can decompose the derivative

$$d = \partial \oplus \bar{\partial} : \Omega^0 \to \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$

Make sure you understand the following simple calculations:

(i) $\partial f = \frac{1}{2}(df - idf \circ J)$ and $\bar{\partial} f = \frac{1}{2}(df + idf \circ J)$

(ii) $f$ is holomorphic if and only if $\bar{\partial} f = 0$

(iii) For a holomorphic chart $(U, z)$ we have: $\partial f = \frac{\partial f}{\partial z} dz$ and $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$ with the usual definition for $\frac{\partial}{\partial z}$ etc.

It will be necessary to also consider bundle valued $k$-forms: if $E \to M$ is an $\mathbb{F}$-vector bundle, then

$$\Omega^0(M, E) := \Gamma(M, E), \quad \Omega^1(M, E) := \Gamma(\text{Hom}_\mathbb{R}(TM, E)) = \Gamma(TM^* \otimes_{\mathbb{R}} E)$$

and etc. If $E \to M$ is a complex vector bundle over a Riemann surface (so each fiber has multiplication by $i$), we can decompose as before

$$\Omega^1(M, E) = \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E) \quad \text{e.g.} \quad \Omega^{0,1}(M, E) = \Gamma(\bar{K} \otimes_{\mathbb{C}} E)$$

In fact, the above case now fits nicely into this more general scheme as $E = \mathbb{C}$. 
Problem 5. Let $M$ be a Riemann surface and $E \to M$ a holomorphic vector bundle. Show that there exists a unique $\mathbb{C}$-linear map ("operator")

$$\bar{\partial}^E : \Gamma(M, E) \to \Omega^{0,1}(M, E)$$

defined locally as follows: if $\{\psi_i\}$ is a local holomorphic frame of $E$ over an open $U \subset M$, then

$$\bar{\partial}^E \left( \sum_{i=1}^r \xi_i \psi_i \right) := \sum_{i=1}^r \bar{\partial} \xi_i \psi_i$$

(more formally could write $\sum_{i=1}^r \bar{\partial} \xi_i \otimes \psi_i$)

for $\xi^i : U \to \mathbb{C}$ smooth functions (note that any smooth section over $U$ can be written in the frame $\{\psi_i\}$ with smooth coefficient functions). Then show that this well-defined operator satisfies

(i) the Leibnitz rule:

$$\bar{\partial}^E (f \psi) = \bar{\partial} f \psi + f \bar{\partial}^E \psi$$

where $f \in C^\infty(M, \mathbb{C})$, and

(ii) for $U \subset M$ open $\ker \bar{\partial}^E|_{\Omega^{0,\infty}(U, \mathbb{C})} = H^0(U, E)$ describes the holomorphic sections. In particular, $H^0(M, E) = \ker \bar{\partial}^E$.

The next problem explains what holomorphic trivialization has to do with solving a delbar equation:

Problem 6. Let $E \to M$ be a holomorphic vector bundle over a Riemann surface $M$ which we assume is smoothly (not necessarily holomorphically) trivializable, say by smooth frames $\{\phi_i : M \to E\}_{1 \leq i \leq r}$. The idea of course is to change the frame $\phi = (\phi_1, \ldots, \phi_r)$ into a holomorphic frame $\psi$ by setting

$$\psi = \phi g$$

for a yet to be determined smooth map $g : M \to \text{Gl}(r, \mathbb{C})$. Show that $\psi$ is a holomorphic frame if and only if $\bar{\partial} g + \alpha g = 0$, where $\alpha \in \Omega^{0,1}(M, \text{gl}(r, \mathbb{C}))$ is given by $\bar{\partial} \phi = \bar{\partial} \phi + \alpha$.

Next we discuss how to prove that such delbar equations can be solved on disks (at least in the rank 1 case; the higher rank "non-abelian case" is harder, of course, and you can see where the difficulties arise).

Problem 7. Let $D \subset \mathbb{C}$ be a centered disk (or open square) and let $g : \bar{D} \to \mathbb{C}$ be a smooth function on the closed disk. Define

$$2\pi i f(z) := \int_D \frac{g(w)}{w - z} dw \wedge d\bar{w}$$

Show that this 2-dimensional integral defines a smooth function $f : D \to \mathbb{C}$ and that $\bar{\partial} f = g$. 
Problem 8. Show that a holomorphic line bundle over any open disk $D \subset \mathbb{C}$ is holomorphically trivializable.

Hint: first show that you can smoothly trivialize (which is a nice problem on its own) and etc.