This class necessarily has to develop some of the basic techniques used in calculus on manifolds. We cannot go through all the detail, so if you are not too familiar with it, Spivak’s little book is a possible reference for differentiable forms, integration, Stokes Theorem etc.

First some comments to the statement “everything one can canonically do to a vector space can be done to a vector bundle”. Here some examples:

**Problem 1.** If $V_i$, $i = 1, 2, 3$ are (always finite dimensional) vector spaces, then we have the vector space of bilinear maps $\text{Bil}(V_1, V_2; V_3)$. Show that given 3 vector bundles $E_i \to M$ one can build the vector bundle $\text{Bil}(E_1, E_2; E_3)$ whose fiber over a point $p \in M$ is $\text{Bil}(E_{1,p}, E_{2,p}; E_{3,p})$ by providing bundle charts. Determine the rank of $\text{Bil}(E_1, E_2; E_3)$.

A bilinear bundle map (with values in $E_3$) is a section $b$ of $\text{Bil}(E_1, E_2; E_3)$. Calculate the expression of $b$ in bundle charts (resp. local frames).

Further show that if $E_i$ are holomorphic bundles (in which case $M$ is always assumed to be a Riemann surface) then $\text{Bil}(E_1, E_2; E_3)$ is a holomorphic bundle.

**Problem 2.** Review the tensor product $V \otimes W$ of two vector spaces over $\mathbb{K}$ (make sure you understand this well). Show that for two vector bundles $E_i \to M$ the tensor product $E_1 \otimes E_2$ (defined fiber wise) is again a vector bundle. What is its rank? If $E_i$ are holomorphic, so will be their tensor product.

Now some examples helping to understand the statement “canonical linear homomorphisms of vector spaces induce canonical bundle homomorphisms”: canonical here means *not depending on choices* such as a basis etc. Even though $V$ and $V^*$ are isomorphic as vector spaces, there is no “canonical” isomorphism (any such isomorphism will need a choice of basis). But there is a canonical isomorphism e.g. between $\text{Hom}(V, W)$ and $V^* \otimes W$, or $\text{Bil}(V_1, V_2; V_3)$ and $V_1^* \otimes V_2^* \otimes V_3$ (write those isomorphisms down!).

**Problem 3.** Let $E, F \to M$ be vector bundles. Show that there is a bundle isomorphism between $\text{Hom}(E, F)$ and $E^* \otimes F$. Furthermore, this isomorphism is holomorphic if $E, F$ are holomorphic bundles.

Note that we could never expect to prove that for a vector bundle $E \to M$ its dual bundle $E^*$ is isomorphic to $E$ (give a counter example). Or, for that matter,
that any bundle is isomorphic to the trivial bundle, i.e., trivializable (note: every vector space of dimension \( r \) is isomorphic to \( \mathbb{K}^r \), but not canonically). I hope this makes the point clear.

**Problem 4.** Show that the set of isomorphisms classes of \( \mathbb{K} \)-line bundles over a manifold is an abelian group under tensor product. What is the inverse, what is the neutral element?

Thus, the set of holomorphic line bundles over a Riemann surface form a group. This is called the **Picard group** \( \text{Pic}(M) \), of \( M \).

**Problem 5.** Show that \( \text{Pic}(\mathbb{C}P^1) \cong \mathbb{Z} \) and give an isomorphism.

We call a manifold \( M \) **orientable** if there is an atlas of charts whose transition functions have positive Jacobi determinant, i.e. \( \det d(\tilde{x} \circ x^{-1}) > 0 \) for all pair of charts \((U, x)\) and \((\tilde{U}, \tilde{x})\). Such an atlas is called oriented.

A **Riemannian metric** on a manifold \( M \) is a section \( g \in \Gamma(\text{Bil}(TM, TM; \mathbb{R})) \), i.e. a bilinear bundle map \( g: TM \times TM \to \mathbb{R} \), such that \( g_p \) is symmetric and \( g_p \) is positive definite for all \( p \in M \). In other words, \( p \mapsto g_p \) is a smoothly varying family of Euclidean inner products. Thus, on a Riemannian manifold one can calculate the length of and angle between vectors in a fixed tangent space, and thus the length of a curve \( L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \, dt \) and the angle between curves. A map \( f: (M, g) \to (N, h) \) between Riemannian manifolds is called **isometric** if \( h(df(v), df(w)) = g(v, w) \) for all \( v, w \in T_pM \) and all \( p \in M \).

A **conformal structure** on a manifold is an equivalence class \([g]\) of Riemannian metrics under the “scaling relation”: \( g \) equivalent to \( h \) if and only if there exists a positive function \( \lambda: M \to \mathbb{R}_+ \) with \( h = \lambda g \). Thus, on a conformal manifold one can calculate angles between vectors (and curves), but length is no longer defined. A map \( f: (M, [g]) \to (N, [h]) \) between conformal manifolds is called **conformal** if \( h(df(v), df(w)) = \rho(p)g(v, w) \) for all \( v, w \in T_pM, p \in M \), for some non-negative function \( \rho: M \to \mathbb{R} \) (note that this condition is independent of the choices of representatives of the conformal classes on \( M \) and \( N \)).

**Problem 6.** Every Riemann surface is orientable and carries a well-defined conformal structure \([g]\).

**Hint:** if \( J: TM \to TM \) is the almost complex structure of the Riemann surface consider the Riemannian metrics for which \( J \) is positive 90 degree rotation in each tangent space.

The converse is nontrivial: an orientable conformal 2-dim real manifold is a Riemann surface. The crux to prove this result is to construct “conformal coordinates/charts”, that is local charts \((U, (x, y))\) in the oriented atlas of \( M \) in which the conformal structure (represented by \( g \)) has the form \( g = \lambda(dx^2 + dy^2) \) for a positive function (depending on the chart of course) \( \lambda: U \to \mathbb{R}_+ \). I hope to get to this proof later in class which requires some linear elliptic analysis.
Next the fundamental relationship between conformality and holomorphicity:

**Problem 7.** Let $f: M \to N$ be a non-constant smooth map between Riemann surfaces. Show that $f$ is holomorphic if and only if $f$ is conformal and preserves the orientation $(\det(d(w \circ f \circ z^{-1})) \geq 0$ for all holomorphic charts $z$ on $M$ and $w$ on $M$).

**Problem 8.** Show that once conformal charts have been proven to exist on an oriented conformal 2-dimensional manifold $(M, [g])$, those then form an atlas whose transition functions are holomorphic, so that $M$ becomes a Riemann surface.

Thus, modulo existence of conformal coordinates, we have shown the equivalence between Riemann surfaces (i.e., 1-dim complex manifolds) and real 2-dim oriented conformal manifolds (those are easy to come by: every oriented smooth surface $M \subset \mathbb{R}^3$ has a conformal structure: restrict the Euclidean inner product of $\mathbb{R}^3$ to each tangent space to get a Riemannian metric on $M$, and take its conformal class; so every smooth oriented surface in $\mathbb{R}^3$ is a Riemann surface!)

**Problem 9.** Recall that a vector field on a manifold is a section $X \in \Gamma(TM)$, i.e., a smooth assignment of tangent vectors to each point of the manifold (as an example take $M \subset \mathbb{R}^m$ open, then $TM = U \times \mathbb{R}^m$ and $X: M \to TM$ is $X(p) = (p, \xi(p))$ for a smooth function $\xi: M \to \mathbb{R}^m$).

An integral curve of $X$ is a smooth map $\gamma: I \to M$ such that $\gamma' = X \circ \gamma$, where $I \subset \mathbb{R}$ is an open interval.

(i) Write down the equation for an integral curve in a chart.
(ii) Show that to every $p_0 \in M$ there exists an integral curve $\gamma: I \to M$ with $0 \in I$ and $\gamma(0) = p_0$. One says, that the integral curve has initial condition $p_0$.
(iii) Show that two integral curves $\gamma_k: I_k \to M$ with $0 \in I_1 \cap I_2$ and $\gamma_1(0) = \gamma_2(0)$ agree on $I_1 \cap I_2$. How would you define the maximal existence interval $I_p \subset \mathbb{R}$ of the integral curve $\gamma_p$ with initial condition $p$?
(iv) Show that if $M$ is compact (or the vector field has compact support) then to any $p \in M$ there is a unique integral curve $\gamma_p: \mathbb{R} \to M$ with initial condition $\gamma_p(0) = p$. Emphasize here is on the fact that $\gamma_p$ is defined on all of $\mathbb{R}$, i.e., for all $p \in M$ the maximal existence interval is $\mathbb{R}$.
(v) Show that for compact $M$ (or the vector field has compact support) the map $\Phi: \mathbb{R} \times M \to M$ given by $\Phi(t, p) := \gamma_p(t)$ is smooth in both variables and satisfies

$$\Phi(t + s, p) = \Phi(t, \Phi(s, p))$$

Deduce that $\Phi_t := \Phi(t, -)$ is a diffeomorphism of $M$ and the map $t \mapsto \Phi_t$ a group homomorphisms of $(\mathbb{R}, +)$ into the diffeomorphism group $\text{Diff}(M)$ of $M$. 

(vi) Let $S \subset M$ be a finite set of points. Show that for any point $p_0 \in M$ and every open neighborhood $U \subset M$ of $p_0$ there exists a diffeomorphism $\varphi: M \to M$ so that $\varphi(S) \subset U$. 