Hw 10 (due April. 29, Thursday)

6.39
[Sol] Since $Y_i \sim \exp(\beta = 1)$, $m_{Y_i}(t) = (1-t)^{-1}$ for $i = 1, 2$. Using Theorem 6.2, we have, for $U = \frac{1}{2}(Y_1 + Y_2)$, $m_U(t) = m_{Y_1}(t/2)m_{Y_2}(t/2) = (1 - t/2)^{-2}$ which is the MGF for a gamma random variable with $\alpha = 2$ and $\beta = 1/2$. Hence $f_U(u) = 4ue^{-2u}$ for $u \geq 0$.

6.42
[Sol] Note that $Y_1 \sim N(5000, 300^2)$, $Y_2 \sim N(4000, 400^2)$ and they are independent. We are interested in $P$ (the elevator will be overloaded) = $P(Y_2 - Y_1 > 0)$. By Theorem 6.3 $Y_2 - Y_1 \sim N(-1000, 500^2)$, as two random variables are independent, $E(Y_2 - Y_1) = 4000 - 5000$ and $V(Y_2 - Y_1) = 300^2 + 400^2$. Let $U = Y_2 - Y_1$ and $Z = \frac{U - E(U)}{\sqrt{V(U)}}$. Then $Z \sim N(0, 1)$ (by Example 6.10) and $P(Y_2 - Y_1 > 0) = P(U > 0) = P(Z > \frac{0 - E(U)}{\sqrt{V(U)}}) = P(Z > \frac{1000}{500}) = P(Z > 2) = .0228$ (Use Table 4).

6.43 (a)-(b)
[Sol] $Y_1, \ldots, Y_n$ are independent, normal random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$.
(a) By Theorem 6.3, $\bar{Y} \sim N(\mu, \sigma^2/n)$.
Then the density function of $\bar{Y}$ is $f(\bar{y}) = \frac{1}{\sqrt{2\pi}(\sigma^2/n)} \exp\left(-\frac{(\bar{y} - \mu)^2}{2(\sigma^2/n)}\right)$.
(b) Since $\bar{Y} \sim N(\mu, \sigma^2/n)$, $Z_{\bar{Y}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ (by Example 6.10). When $\sigma^2 = 16$ and $n = 25$, $P(\bar{Y} - \mu \leq 1) = P\left(\left|\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right| \leq \frac{1}{4/5}\right) = P(|Z| \leq \frac{5}{4}) = 0.7888$ (Use Table 4).

6.55
[Sol] Let $Y = Y_1 + Y_2$ with $Y_1, Y_2$ the number that arrive in each hour. Since $Y_1 \sim \text{poi}(7)$ and $Y_2 \sim \text{poi}(7)$, $Y = Y_1 + Y_2 \sim \text{poi}(14)$ (by exercise in class). Then $P(Y > 20) = 1 - P(Y \leq 19) = 1 - \sum_{y=0}^{19} e^{-14}14^y/y! = .077$.

6.2
[Sol]
(a) $F_{U_1}(u) = P(U_1 \leq u) = P(3Y \leq u) = P(Y \leq u/3) = F_Y(u/3)$. Then
$$F_{U_1}(u) = F_Y(u/3) = 0 \quad \text{for} \quad u/3 < -1,$$
$$= \int_{-1}^{u/3} (3/2)y^2 dy = \frac{1}{2} \left(\frac{u^3}{27} + 1\right) \quad \text{for} \quad -1 \leq u/3 < 1,$$
$$= 1 \quad \text{for} \quad u/3 \geq 1.$$
I.E.,

\[ F_{U_1}(u) = \begin{cases} 
0 & \text{for } u < -3, \\
\frac{1}{2} \left( \frac{u^3}{27} + 1 \right) & \text{for } -3 \leq u < 3, \\
1 & \text{for } u \geq 3.
\end{cases} \]

Then the PDF for \( U_1 \) is

\[ f_{U_1}(u) = \begin{cases} 
\frac{u^2}{18} & \text{for } -3 < u < 3, \\
0 & \text{elsewhere}.
\end{cases} \]

(b) \( F_{U_2}(u) = P(U_2 \leq u) = P(3 - Y \leq u) = P(Y \geq 3 - u) = 1 - F_Y(3 - u). \)

Since

\[ F_Y(3 - u) = \begin{cases} 
0 & \text{for } 3 - u < -1, \\
\frac{1}{2} ((3 - u)^3 + 1) & \text{for } -1 \leq 3 - u < 1, \\
1 & \text{for } 3 - u \geq 1.
\end{cases} \]

Then,

\[ F_{U_2}(u) = 1 - F_Y(3 - u) = \begin{cases} 
1 - \frac{1}{2} ((3 - u)^3 + 1) & \text{for } 2 < u \leq 4, \\
1 & \text{for } u > 4,
\end{cases} \]

\[ = 1 - \frac{1}{2} ((3 - u)^3 + 1) & \text{for } 2 < u \leq 4,
\]

\[ = 0 & \text{for } u \leq 2.
\]

Then the PDF for \( U_2 \) is

\[ f_{U_2}(u) = \begin{cases} \frac{3}{2} (3 - u)^2 & \text{for } 2 < u < 4, \\
0 & \text{elsewhere}.
\end{cases} \]

(c) \( F_{U_3}(u) = P(U_3 \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u}). \) Then

\[ F_{U_3}(u) = P(-\sqrt{u} \leq Y \leq \sqrt{u}) = \begin{cases} 
0 & \text{for } \sqrt{u} < 0, \\
\int_{-\sqrt{u}}^{\sqrt{u}} (3/2)y^2 dy = u\sqrt{u} & \text{for } 0 \leq \sqrt{u} < 1, \\
1 & \text{for } \sqrt{u} \geq 1.
\end{cases} \]
I.E.,

\[ F_{U_3}(u) = \begin{cases} 
0 & \text{for } u < 0, \\
 \frac{u}{\sqrt{u}} & \text{for } 0 \leq u < 1, \\
1 & \text{for } u \geq 1.
\end{cases} \]

Then the PDF for \( U_3 \) is

\[ f_{U_3}(u) = \begin{cases} 
\frac{3}{2} \sqrt{u} & \text{for } 0 < u < 1, \\
0 & \text{elsewhere.}
\end{cases} \]

7.9

(a) \( P(|\bar{Y} - \mu| \leq .3) = P[-.3 \leq \bar{Y} - \mu \leq .3] = P\left( \frac{-3}{\sigma/\sqrt{n}} \leq \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{3}{\sigma/\sqrt{n}} \right) = P(-1.2 \leq Z \leq 1.2) = 1 - 2P(Z > 1.2) = 1 - 2(0.1151) = .7698. \)

(b) \( P(|\bar{Y} - \mu| \leq .3) = P[-.3 \leq \bar{Y} - \mu \leq .3] = P\left( \frac{-3}{\sigma/\sqrt{n}} \leq \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{3}{\sigma/\sqrt{n}} \right) = P(-.3\sqrt{n} \leq Z \leq .3\sqrt{n}) = 1 - 2P(Z > .3\sqrt{n}). \)

For \( n = 25 \), we have \( P(|\bar{Y} - \mu| \leq .3) = 1 - 2P(Z > 1.5) = .8664. \)

For \( n = 36 \), we have \( P(|\bar{Y} - \mu| \leq .3) = 1 - 2P(Z > 1.8) = .9284. \)

For \( n = 49 \), we have \( P(|\bar{Y} - \mu| \leq .3) = 1 - 2P(Z > 2.1) = .9642. \)

For \( n = 64 \), we have \( P(|\bar{Y} - \mu| \leq .3) = 1 - 2P(Z > 2.4) = .9836. \)

7.11 Since the distribution of basal areas is normally distributed with mean \( \mu \) and variance \( \sigma^2 = 16 \), \( \bar{Y} \) will also be normally distributed from Theorem 7.1. Then \( P(|\bar{Y} - \mu| \leq 2) = P[-2 \leq \bar{Y} - \mu \leq 2] = P\left( \frac{-2}{\sigma/\sqrt{n}} \leq \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{2}{\sigma/\sqrt{n}} \right) = P(-1.5 \leq Z \leq 1.5) = 1 - 2P(Z > 1.5) = 1 - 2(.0668) = .8664. \)

7.12 It is necessary to have \( .90 = P(|\bar{Y} - \mu| \leq 1) = P\left( \frac{-\sqrt{4}}{4} \leq Z \leq \frac{-\sqrt{4}}{4} \right). \) Since \( Z \sim N(0, 1) \) is symmetric at 0, this inequality will be satisfied if we take \( \sqrt{n}/4 = 1.645 \) or \( n = 43.30. \) Hence 44 trees must be sampled.