Show all work. Explain your answers. Partial credits will be given.

1. The content of a can of coke produced by Virgin Cola can be modeled as a random variable \( X \sim N(\mu, \sigma^2) \). The declared content of a can is 33 cl and, when this happens, the process is declared in control. Inspections are regularly carried out to check that neither the content exceeds the declared value 33 cl, thus producing a loss for the company, nor is below the declared value 33 cl, thus creating customer dissatisfaction. When either of these situations happens, the process is declared out of control.

a) Suppose the process is in control, so that the mean content is \( \mu = 33 \). Suppose also that you know that \( \sigma = 2 \), in the population. Compute the probability that the content of a can chosen at random exceeds 33.5 cl.

b) Suppose that the process is in control, so that the mean content is \( \mu = 33 \), and that you know that \( \sigma = 2 \), in the population. Compute the probability the average content of a sample of 20 cans exceeds 33.5 cl.

c) Suppose the following quality control process is proposed: “sample 20 cans and if the average value exceeds 33.5 cl or is below 32.5 cl, then declare the process out of control”. Compute the probability of declaring the process out of control when it is not.

2. In a state lottery, a player selects a three digit number and wins 500 dollars if that number is drawn from a box containing tickets labeled 000, \ldots, 999. If it costs 1 to play the game, what is the player’s expected gain? Now suppose that 1000 people play the game each day. Let \( S \) denote the state’s total gain in a day. Find a value \( c \) for which \( P(|S - ES| < c) \) is approximately 0.95.

3. Suppose that \( X \) and \( Y \) have joint density \( f(x, y) = x + y \) for \( 0 \leq x, y \leq 1 \) and \( f(x, y) = 0 \) otherwise. Find

a) \( P[Y < \frac{1}{2} | X = \frac{1}{2}] \).

b) Are \( X \) and \( Y \) independent? Explain.

c) \( E(Y | X = \frac{1}{2}) \).

d) \( \text{Cov}(X, Y) \)

e) \( \text{Var}(2X + 3Y) \)

f) \( \text{Cov}(2X + 3Y, X - Y) \)

g) The pdf of \( 2Y + 1 \)

4. Let \( Y_1, Y_2, \ldots, Y_n \) be independent, uniformly distributed random variables on the interval \([0, \theta]\). Find

a) probability distribution function of \( U = \max(Y_1, Y_2, \ldots, Y_n) \).

b) density function of \( U \).

c) mean and variance of \( U \).

5. The National Fire Incident Reporting Service stated that, among residential fires, 73% are in family homes, 20% are in apartments, and 7% are in other types of dwellings.

a) If four residential fires are independently reported on a single day, what is the probability that two are in family home, one is in apartment, and one is in another type of dwelling?

b) The typical cost of damages caused by a fire in a family home is $20,000. Comparable costs for an apartment fire and for fire in other dwelling types are $10,000 and $2000, respectively. If four fires are independently reported, find the expected total damage cost and the variance of the total damage cost.