Math 331 - ODE **Due: 11 or 12 December, 2018**

Written Problem Set # 3

Attempt and turn in all problems.

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1. Write each of the following differential equations as a first order system. Is the system linear or nonlinear?

   (a) $y''' + 4t^2 y'' - y' = 0$

   (b) $y'' + y' - y^2 = 0$

   (c) $y'''' - 2y = 5e^{-t}$

2. For each of the following matrices, compute the eigenvalues and eigenvectors. Also indicate the type of the critical point of the system (saddle, source or sink, spiral, etc.)

   (a) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$; (b) $B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$; (c) $C = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix}$.

3. Do the following computations.

   (a) Let
   $$A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix},$$
   and compute the following:
   (i) $A - 2B$; (ii) $3A + B$; (iii) AB; (iv) BA; (v) A^*; (vi) B^*.

   (b) Let
   $$\mathbf{x} = \begin{pmatrix} 2, & 3i, & 1-i \end{pmatrix}^T \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} -1+i, & 2, & 3-i \end{pmatrix}^T,$$
   and compute the following, showing details:
   (i) $\mathbf{x}^T\mathbf{y}$; (ii) $\mathbf{y}^T\mathbf{y}$; (iii) $(\mathbf{x}, \mathbf{y})$; (iv) $(\mathbf{y}, \mathbf{y})$.

4. Consider the system
   $$\mathbf{x}' = A\,\mathbf{x}, \quad \text{with} \quad A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix},$$
   and the two vector-valued functions
   $$\mathbf{x}_1(t) = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix},$$

   (a) Show that both $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the system.

   (b) Show that for any constants c_1 and c_2, $\mathbf{x} = c_1\,\mathbf{x}_1 + c_2\,\mathbf{x}_2$ is also a solution.

   (c) Find the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

   (d) Show that the pair $(\mathbf{x}_1,\ \mathbf{x}_2)$ forms a fundamental set of solutions of the given system.

   (e) Find the solution that satisfies the initial condition $\mathbf{x}(0) = (1, 2)^T$.

5. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

and describe the behavior of the solution as $t \to \infty$.

6. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

and describe the behavior of the solution as $t \to \infty$.

7. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and describe the behavior of the solution as $t \to \infty$.

8. Consider the system with (real) parameter α,

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}.$$

   (a) Determine the eigenvalues in terms of α.
   (b) Find the bifurcation value or values of α; these are the values at which the qualitative nature of solutions change.
   (c) For each such bifurcation value α_c, sketch a phase portrait for values of α slightly below, equal to and slightly above α_c. Be sure to indicate the direction of flow.