Math 331 - ODE **Due: 27 or 28 November, 2018**

Written Problem Set # 2

Attempt and turn in all problems.
A table of Laplace transforms is attached.

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1. For each of the following functions do the following:
   (i) Write the function as a piecewise function and sketch its graph;
   (ii) Write the function as a combination of terms of the form $u_a(t)\,k(t-a)$; and
   (iii) compute the Laplace transform.

   (a) $f(t) = t(1 - u_1(t)) + e^t(u_1(t) - u_2(t))$
   (b) $g(t) = \sin(2t) + u_\pi(t)(t/\pi - \sin(2t)) + u_{2\pi}(t)(2\pi - t)/\pi$
   (c) $h(t) = u_0(t) + \sum_{k=1}^{5}(-1)^k u_k(t)$

2. Solve the initial value problem

$$y' + 6y = g(t), \quad \text{where} \quad g(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 12 & \text{if } 1 \le t < 7 \\ 0 & \text{if } 7 \le t \end{cases},$$

   and with $y(0) = 4$.

3. Solve the initial value problem

$$y'' + y = g(t) \quad \text{where} \quad g(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t \end{cases},$$

   and with $y(0) = 0$ and $y'(0) = 0$.

4. Consider the mass-spring system described by the initial value problem

$$y'' + 4y = \sin t + u_{\pi/2}(t)\cos t \quad \text{with} \quad y(0) = 0,\ y'(0) = 0.$$

   Find the solution of the initial value problem.
   Hint: Recall that $\cos t$ is a shift of $\sin t$.

5. Determine the value of the following integrals:

   (a) $\displaystyle\int_2^7 \delta(t+1)\,dt$;

   (b) $\displaystyle\int_{-2}^7 \delta(t-1)\,dt$;

   (c) $\displaystyle\int_{-2}^4 (2t^4 - t^3 + 7t^2 - 1)\,\delta(t-1))\,dt$;

   (d) $\displaystyle\int_1^8 \ln(t^2)\,\delta(t-e)\,dt$.

6. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi), \quad \text{with ICs} \quad y(0) = 1, \quad y'(0) = 0,$$

and sketch a graph of the solution.

7. Solve the initial value problem

$$y'' + y = \delta(t - 2\pi)\, \cos t, \quad y(0) = 0, \quad y'(0) = 1,$$

and sketch a graph of the solution.

8. Consider the initial value problem

$$y'' + 2y' + y = k\, \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0,$$

where k is a constant parameter.

(a) Find the solution of the initial value problem.

(b) Find the value of k for which the solution has a maximum value of 2.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n$ a positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
5. $\sin(at)$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos(at)$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 5
7. $\sinh(at)$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
8. $\cosh(at)$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 6
9. $e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 10
10. $e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 11
11. $t^n e^{at}, \quad n$ a positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 14
12. $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 17
16. $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 21