Math 331 - ODE **Due: 3 October 2018**

Written Problem Set # 1

These are practice problems for the midterm exam.
You should attempt all of them, but **turn in only the even-numbered problems**!

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1. Solve the initial value problem
$$\frac{dy}{dt} = y^2\,[t + \cos(t)], \quad \text{with} \quad y(0) = 6.$$

2. Solve the initial value problem
$$y' = \frac{3x\sin(x^2) - 1}{3 + 2y}, \quad \text{with} \quad y(0) = 2.$$

3. Mary initially deposits \$1000 in a savings account that pays interest at the rate of 5% per year (compounded continuously). She also arranges for \$25 per week to be deposited automatically into her account.
   (a) Assume that weekly deposits can be approximated by continuous deposits. Write down an initial value problem for her account balance $S(t)$ over time, where t is measured in years.
   (b) How long does she need to save for to buy a \$5000 car with cash?

4. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to one tenth of its original mass.

5. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

6. At what yearly rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

7. Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature T_m, the rate of change of T at time t is proportional to $T(t) - T_m$, so that T satisfies a differential equation of the form
$$T' = -k\,(T - T_m).$$
Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the temperature decay constant of the medium.
   (a) A thermometer is moved from a room where the temperature is $70°F$ to a freezer where the temperature is $12°F$. After 30 seconds the thermometer reads $40°F$. What does it read after 2 minutes?
   (b) An object is placed in a room where the temperature is $20°C$. The temperature of the object drops by $5°C$ in 4 minutes and by $7°C$ in 8 minutes. What was the temperature of the object when it was initially placed in the room?

8. Consider the equation for a certain population $P(t)$,

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1).$$

(a) Find all the equilibrium solutions of this equation. Draw the phase line and determine the stability of each equilibrium.

(b) Draw a rough graph of the solutions with each of the initial conditions

$$P(0) = 1/4, \quad P(0) = 3/2, \quad \text{and} \quad P(0) = 3.$$

(c) Due to a series of unfortunate events, a certain fraction α of the population dies every year, so the equation becomes

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1) - \alpha P.$$

Without solving the equation, analyze the system for each of the values $\alpha = 0.2$, $\alpha = 0.25$ and $\alpha = 0.3$, and determine the maximum value of α for which a large population will not become extinct (this is called a *bifurcation point*).

9. Solve the initial value problem

$$\frac{dy}{dt} = -3\frac{y}{t} - 2 - t^{-4}, \qquad y(1) = 4.$$

10. Solve the initial value problem

$$\frac{dy}{dt} = \frac{-e^t}{y}, \qquad y(0) = -2.$$

11. A home buyer can afford to spend no more than \$1000 per month on mortgage payments. Suppose that the interest rate is 5% (per year) and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow.

(b) Determine the total interest paid during the term of the mortgage.

12. Solve the initial value problem

$$\frac{dy}{dt} = y\,t + 2t, \qquad y(3) = 2.$$

13. Solve the initial value problem

$$\frac{dy}{dt} = 9\,y + e^{-3t}, \qquad y(0) = 3.$$

14. Solve the initial value problem

$$\frac{dy}{dt} = 6y + 2t - 4, \qquad y(0) = 3.$$

15. Solve the initial value problem

$$\frac{dy}{dx} = \frac{1+y^2}{y\,e^{-x}}, \qquad y(0) = -2.$$

16. Solve the initial value problem

$$\frac{dy}{dx} = 1 + y^2, \qquad y(0) = -2.$$

17. Solve the initial value problem

$$\frac{dy}{dx} = y\,(y+2), \qquad y(0) = 1.$$

18. Find the general solution of

$$x^2 + y^2 + 2\,x\,y\,y' = 0.$$

19. Solve the initial value problem

$$\big(\sin(x) - y\,\sin(x) - 2\,\cos(x)\big) + \cos(x)\,y' = 0, \qquad y(0) = -1.$$

20. Find the general solution of

$$x\,y^2 + 2\,x\,y\,y' = 0.$$

21. By making an appropriate change of variable, find the general solution of

$$y' = \frac{y}{x} + e^{-y/x}.$$

22. By making an appropriate change of variable, find the general solution of

$$\frac{dy}{dx} - y = x\,y^2.$$

23. Find *all* functions $M(x,y)$ such that the equation

$$M(x,y)\;dx + (x^2 - y^2)\;dy = 0$$

is exact.

24. A tank initially contains a solution of $30g$ of dye in 60ℓ of water. Water containing $8g$ of dye per liter is added to the tank at $20\ell/min$, and the resulting solution is drained at the same rate. Find the quantity $Q(t)$ of dye in the tank at time $t > 0$.

25. Consider the differential equation

$$\frac{dy}{dt} = 3\,y^3 - 12\,y.$$

(a) Find the equilibrium solutions, draw the phase line, and identify the stability of each equilibrium.

(b) Sketch the solutions corresponding to the initial conditions

$$y(0) = 2, \quad y(0) = \frac{3}{2}, \quad \text{and} \quad y(0) = -1.$$