DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 331 **Final Exam** **Fall 2017**

Name: ______________________________ **Student ID Number:** _______________

Instructor name: ___________________________ **Your section number** ________

In this exam there are 11 pages, including this one, and there are 6 problems. The last two pages consist of a table of Laplace transform and a blank page you can use for computations. Feel free to take them off.

1.	(20)	________
2.	(15)	________
3.	(15)	________
4.	(15)	________
5.	(20)	________
6.	(15)	________
Total	(100)	________

Instructions:

- You must explain how you arrived at your answers, and show your algebraic calculations.
- You can leave fractions and square roots in your answers – no need to give decimal expansions.

1. (20 points)

(a) (5 points) Find the general solution of the second-order equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0.$$

(b) (5 points) Find the general solution of the second-order equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = t.$$

(c) (5 points) Find the general solution of the second-order equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 3e^{2t}.$$

(d) (5 points) Find the solution of initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 3e^{2t}\,, \quad y(0) = 0, y'(0) = 1\,.$$

2. (15points) For this problem, you do not need to solve any differential equation explicitly. But you should explain how you get to your answer.

   (a) (5 points) An object of mass $m = 2\,lb$ stretches a spring 6 inches in equilibrium and there is no friction. (Recall $g = 32\,ft/sec^2$). What is the period T of a typical solution $y(t)$ for the motion of the spring-mass system?

   Period $T =$

   (b) (5 points) An object of mass $m = 2\,lb$ stretches a spring 6 inches in equilibrium and there is no friction. The system is subjected to an external force equal to $F(t) = 7\cos(\omega t)$. For which value of ω does the system exhibits resonances?

   Resonant frequency $\omega =$

   (c) (5 points) An object of mass $m = 2\,lb$ stretches a spring 6 inches in equilibrium and there is friction with a friction constant $c \geq 0$. For which range of the values of c does the motion of the spring- mass system exhibit oscillations (damped or not)?

   Oscillations when c satisfies

3. (15 points) Consider the matrix $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ and the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

(a) (10 points) Find the eigenvalues, eigenvectors and draw the phase portrait of the system indicating clearly the eigenvectors if applicable.

Eigenvalues:

Eigenvectors:

Phase portrait:

(b) (5 points) Solve the initial value problem $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

4. (15 points) For the following systems $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ compute the eigenvalues and determine their types: source, saddle, sink (node), center, spiral source, spiral sink?

(a) (5 points) $A = \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$.

(b) (5 points) $A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$.

(c) (5 points) $A = \begin{pmatrix} -2 & 3 \\ 7 & 2 \end{pmatrix}$.

5. (20 points) Using the Laplace transform solve the initial value problem $\frac{d^2y}{dt^2} + 4y = g(t)$ with $y(0) = 1, y'(0) = 0$ and

$$g(t) = \begin{cases} -2 & \text{if } 0 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases} .$$

6. (15 points) Solve the initial value problem $\frac{d^2y}{dt} + 9\frac{dy}{dt} + 14y = \delta(t-8)$ with initial conditions $y(0) = 0, y'(0) = 2$ using the Laplace transform and make a graph of the solution.

Graph of the solution (2 points):

Table of Laplace transforms

$f(t)$	$\mathcal{L}(f(t))$		$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$			
t	$\frac{1}{s^2}$			Derivatives
t^2	$\frac{2}{s^3}$		y	$\mathcal{L}(y)$
t^n	$\frac{n!}{s^{n+1}}$		y'	$s\mathcal{L}(y) - y(0)$
e^{at}	$\frac{1}{s-a}$		y''	$s^2\mathcal{L}(y) - sy(0) - y'(0)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$			
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$			
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$			t-Shift
$\cosh(at)$	$\frac{s}{s^2-a^2}$		$f(t)$	$F(s)$
$\sinh(at)$	$\frac{a}{s^2-a^2}$		$u_a(t)f(t-a)$	$e^{-as}F(s)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$			
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$			s-Shift
$\delta(t-a)$	e^{-as}		$f(t)$	$F(s)$
$u_a(t)$	$\frac{e^{-as}}{s}$		$e^{at}f(t)$	$F(s-a)$