**Math 624** **Spring 2019**

# Analysis 2

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**Homework # 4: due Thursday, 4/11/19**

1. (a) Show that a (complex) normed vector space is an inner product space if and only if the paralellogram law holds:

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$

[Hint: you need to find both real and imaginary parts of $(x, y)$.]

(b) Show that $L^p[0,1]$ can be realized as a Hilbert space only if $p = 2$.

2. H-N, page 144, exercise 6.2.

3. Given an independent set $U = \{u_\alpha\}$ in a separable Hilbert space, use the Gram-Schmidt procedure to show that there is an orthonormal basis $V$ such that $Span(U) = Span(V)$.

4. H-N, page 145, exercise 6.8.

5. H-N, page 145, exercise 6.9.

6. H-N, page 146, exercise 6.13.

7. Let $H = L^2([\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle, with inner product

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\, \overline{G(e^{i\theta})}\, d\theta.$$

Using the mapping

$$\phi : \mathbb{R} \to S^1 \quad \text{given by} \quad \phi(x) = \frac{i-x}{i+x}$$

of $\mathbb{R}$ to the unit circle, show that:

(a) The map $U : H \to L^2(\mathbb{R})$ given by

$$U(F) = f, \quad \text{where} \quad f(x) = \frac{1}{\sqrt{\pi}(i+x)}\, F \circ \phi(x)$$

is unitary.

(b) As a result,

$$\left\{ \frac{1}{\sqrt{\pi}} \left( \frac{i-x}{i+x} \right)^n \frac{1}{i+x} \right\}_{n \in \mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.