Math 624 **Spring 2019**

Analysis 2

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Homework # 3: due Tuesday, 4/2/19

1. Show that a linear functional on a vector space X is bounded iff its kernel is closed.
2. Let X be a Banach space and let $T \in B(X, X)$.
   (a) If I is the identity operator and $\|I - T\| < 1$, show that T is invertible (use a Neumann series).
   (b) If T is invertible and $\|S - T\| < 1/\|T^{-1}\|$, show that S is invertible. Thus the set of invertible operators is open in $B(X, X)$.
3. We define *quotient spaces* as follows. If M is a closed subspace of a vector space X, say $x \sim y$ iff $x - y \in M$. Then $\sim$ is an equivalence relation and we denote the equivalence of x by $x + M$ and the set of these by $X/M = \{x + M \mid x \in X\}$.
   (a) Define linear operations on X/M and norm $\|x + M\| = \inf\{\|x + y\| \mid y \in M\}$, so that X/M becomes a normed vector space. Moreover, if X is Banach, so is X/M.
   (b) Given $\epsilon > 0$, there exists $x \in X$ with $\|x\| = 1$, such that $\|x + M\| \geq 1 - \epsilon$. Also, the projection $\pi : X \to X/M$ given by $\pi(x) = x + M$ has norm 1.
   (c) If $\|\cdot\|$ is a seminorm (ie $\|x\| = 0$ for $x \neq 0$ is allowed), then this construction for $M = \{x \in X \mid \|x\| = 0\}$ turns X/M into a normed vector space.
4. For a convex set $K \subset X$, we defined the Minkowski gauge function by
   $$p_K(x) = \inf\big\{r \geq 0 \mid x \in r\,K\big\}.$$
   Show that p_K satisfies the following conditions:
   (a) $p_K(t\,x) = t\,p_K(x)$ for any $t \geq 0$;
   (b) $p_K\big(\alpha x + (1-\alpha)y\big) \leq \alpha\, p_K(x) + (1-\alpha)\, p_K(y)$ for any $\alpha \in [0.1]$;
   (c) $p_K(x) \leq 1$ for $x \in K$ and $p_K(x) \geq 1$ for $x \notin K$.
5. Prove that given a subspce Z of a normed vector space X and $y \in X$ with $dist(y, Z) = \delta$, there exists $\Lambda \in X^*$ satisfying
   $$\|\Lambda\| \leq 1, \quad \Lambda(y) = \delta, \quad \text{and} \quad \Lambda(z) = 0 \quad \text{for all} \quad z \in Z.$$
6. Let X be a vector space and $P \subset X$ such that: (i) if $x, y \in P$ then $x + y \in P$; (ii) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$; and (iii) if $x \in P$ and $-x \in P$, then $x = 0$. Check that the relation $\leq$ defined by $x \leq y$ iff $y - x \in P$ defines a partial ordering on X. Next, prove the *Krein Extension Theorem*: Suppose that M is a subspace of X, such that for each $x \in X$, there is a $y \in M$ satisfying $x \leq y$, and f is a linear functional on M such that $f(x) \geq 0$ for $x \in M \cap P$. Then there is a linear functional F on all of X such that $F(x) \geq 0$ for all $x \in P$, and $F\big|_M = f$. [Hint: consider $p(x) = \inf\big\{f(y) \mid y \in M,\ x \leq y\big\}$.]