Math 624 **Spring 2019**

Analysis 2

Robin Young

Homework # 2: due Tuesday, 3/26/19

1. H-N, page 59, exercise 2.7.
2. H-N, page 59, exercise 2.9.
3. We briefly analyze the Babylonian (or Greek?!) method for approximating square roots.
   (a) Apply Newton's method to the function $f(x) = x^2 - a$, to get the iteration
   $$x_{k+1} = Tx_k, \quad \text{where} \quad Tx = \frac{1}{2}\big(x + \frac{a}{x}\big),$$
   and T is regarded as a map on $(0, \infty)$. Clearly $\sqrt{a}$ is the unique fixed point.
   (b) Show that T is contractive on $[k\sqrt{a}, \infty)$ if and only if $k > 1/\sqrt{3}$, and calculate the corresponding contraction constant ρ.
   (c) Show that for any $x > 0$, $Tx - \sqrt{a} \geq 0$, so we can take $k = 1$, for which ρ is minimized. Note that $Tx_1 - Tx_2 = o(x_1 - x_2)$ near $\sqrt{a}$, so we expect superlinear convergence.
   (d) Setting $e_k = x_k - \sqrt{a}$, show by induction that
   $$\frac{e_{k+1}}{2\sqrt{a}} \leq \Big(\frac{e_1}{2\sqrt{a}}\Big)^{2^k}.$$
   This is *quadratic convergence*, typical for (convergent) Newton methods.
4. H-N, page 78, exercise 3.3.
5. H-N, page 79, exercise 3.7.
6. Let X and Y be normed linear spaces, and denote the space of bounded maps from X to Y by $B(X, Y)$. Show that $B(X, Y)$ is a normed linear space and that if Y is complete, the so is $B(X, Y)$.
7. Prove Young's inequality: for a, $b > 0$,
   $$a\,b \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{provided} \quad \frac{1}{p} + \frac{1}{q} = 1,$$
   and use this to reprove Hölder's inequality.
   [Hint: For Young, use Jensen's inequality with $\log t$; then choose $a = f(x)/A$, $b = g(x)/B$ for appropriate scalings.]
8. H-N, page 121, exercise 5.3.
9. H-N, page 122, exercise 5.9.
10. H-N, page 123, exercise 5.16.