Math 624 **Spring 2019**

Analysis 2

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Homework # 1: due Tuesday, 2/19/19

1. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$, given by
$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$
Compute the integrals
$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx,$$
and explain in the context of Fubini's theorem.

2. Show that the following three definitions of continuity at $x \in X$ of a map $f : X \to Y$ of metric spaces are equivalent:
   - Given $\epsilon > 0$, there exists $\delta > 0$ such that if $d_X(x,z) < \delta$, then $d_Y\big(f(x), f(z)\big) < \epsilon$;
   - For any sequence $x_n \to x$, we have $f(x_n) \to f(x)$;
   - For any open set $G \subset Y$ containing $f(x)$, the pre-image $f^{-1}(G) \subset X$ is open.

3. Carry out the construction of the completion of a metric space X:
   (a) Define the relation $\sim$ of the set $\mathbf{C} = \big\{\langle x^k \rangle\big\}$ of Cauchy sequences in X by
   $$\{x^k\} \sim \{y^k\} \quad \text{iff} \quad \lim_{k\to\infty} d(x^k, y^k) = 0,$$
   and check it's an equivalence relation. Set $\widetilde{X} = \mathbf{C}/\!\sim$, the set of equivalence classes.
   (b) Define $\widetilde{d}$ on $\widetilde{X}$ by
   $$\widetilde{d}\big(\langle x^k \rangle, \langle y^k \rangle\big) = \lim_{k\to\infty} d(x^k, y^k),$$
   and show that $\widetilde{d}$ is well-defined and a metric.
   (c) Define the map $\iota : X \to \widetilde{X}$ by $\iota(x) = \langle x \rangle$, the constant sequence, and show that ι is an isometry. Moreover, the image $\iota(X)$ is dense in $\widetilde{X}$.
   (d) Use a diagonal argument to show that $\widetilde{X}$ is complete.

   You may skip some details as appropriate, but try to avoid reference to the text!

4. Show that Bolzano-Weierstrass (every bounded sequence has a convergent subsequence) implies Heine-Borel (a set is sequentially compact iff it is closed and bounded) in $\mathbb{R}^d$ (d finite!).

5. H-N, page 32, exercise 1.16.

6. H-N, page 33, exercise 1.20.

7. Recall the heat kernel

$$K_h(x) = \frac{1}{h\sqrt{\pi}}\, e^{-x^2/h}$$

is a good kernel on $\mathbb{R}$ and that $K_h * f \to f$ uniformly for any f supported in $[-R, R]$. For such f, we can write

$$K_h * f(x) = \frac{1}{h\sqrt{\pi}} \int_{-R}^{R} f(u) e^{-(x-u)^2/h}\, du.$$

Use the approximation $e^t \approx \sum_{k=0}^{N} t^k/k!$, uniform on compact sets, to approximate $K_h * f$ by a polynomial of degree $2N$. Now given fixed interval $[a, b]$, choose R such that $[a, b] \subset (-R, R)$ and find a continuous $\widetilde{f} : [-R, R]$ which extends f. Then the restriction of $K_h * \widetilde{f}$ to $[a, b]$ is a uniform polynomial approximation of f on $[a, b]$. This is essentially Weierstrass' proof of the polynomial approximation theorem.