Section 4.2: Linear Transformations and Isomorphisms

**Definition:** Let $V$ and $W$ be any two vector spaces. A function $T : V \rightarrow W$ is called a linear transformation if $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(kv) = kT(v)$ for all $v, v_1, v_2 \in V$ and $k \in \mathbb{R}$. If $T : V \rightarrow W$ is a linear transformation we define the image and kernel of $T$ by:

$$\text{im}(T) = \{w \in W : w = T(v) \text{ for some } v \in V\}, \quad \ker(T) = \{v \in V : T(v) = 0\}.$$  

Note that $\text{im}(T)$ is a subspace of $W$ and $\ker(T)$ is a subspace of $V$.

**Definition:** If $\text{im}(T)$ is finite dimensional, then the rank of $T$ is defined to be $\text{rank}(T) = \dim(\text{im}(T))$. If $\ker(T)$ is finite dimensional, the nullity of $T$ is defined to be $\text{nullity}(T) = \dim(\ker(T))$.

**Fact:** Let $T : V \rightarrow W$ be a linear transformation between two vector spaces $V$ and $W$. If $V$ is finite-dimensional, then the rank-nullity theorem holds:

**Ex 1:** $C^\infty$ is the subset of $F(\mathbb{R}, \mathbb{R})$ containing all functions that can be differentiated infinitely many times. Let $D : C^\infty \rightarrow C^\infty$ be defined by $D(f) = f'$. Show that $D$ is a linear transformation. Describe the image and kernel of $D$.

**Ex 2:** Let $C[0,1]$ be the vector space of all continuous functions $f : [0,1] \rightarrow \mathbb{R}$. Define $I : C[0,1] \rightarrow \mathbb{R}$ by $I(f) = \int_0^1 f(x) \, dx$. Show that $I$ is a linear transformation. Describe the image and kernel of $I$.

**Ex 3:** Let $V$ be the infinite-dimensional vector space of all infinite sequences of real numbers. Define $T : V \rightarrow V$ by $T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_2, \ldots)$. Show that $T$ is a linear transformation. Describe the image and kernel of $T$.  

1
Ex 4: Define a linear transformation $L : M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}^4$ by

$$L \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Show that $L$ is invertible.

**Definition:** An invertible linear transformation $T$ is called an isomorphism. We say that a vector space $V$ is isomorphic to $W$ if there exists an isomorphism $T : V \to W$.

**Note:** Isomorphisms of vector spaces forms an equivalence relation on the set of vector spaces; two vector spaces are isomorphic if they are essentially the same vector space.

**Theorem:** If $B = \{v_1, \ldots, v_n\}$ is a basis of a vector space $V$, then the coordinate transformation $L_B(v) = [v]_B$ from $V$ to $\mathbb{R}^n$ is an isomorphism. (This means that $V$ is isomorphic to $\mathbb{R}^n$).

**Fact:** Every $n$-dimensional vector space $V$ is isomorphic to $\mathbb{R}^n$!

Ex 5: Let $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that the transformation $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ defined by $T(A) = SAS^{-1}$ is an isomorphism.

**Properties:** Let $V$ and $W$ be finite-dimensional vector spaces.

(a) A linear transformation $T : V \to W$ is an isomorphism if and only if $\ker(T) = \{0\}$ and $\text{im}(T) = W$. (This holds even if $V$ and $W$ are not finite-dimensional).

(b) If $V$ is isomorphic to $W$, then $\dim(V) = \dim(W)$.

(c) Suppose $T : V \to W$ is a linear transformation with $\ker(T) = \{0\}$. If $\dim(V) = \dim(W)$, then $T$ is an isomorphism.

(d) Suppose $T : V \to W$ is a linear transformation with $\text{im}(T) = W$. If $\dim(V) = \dim(W)$, then $T$ is an isomorphism.
Ex: Define $L : P_n \to \mathbb{R}^3$ by

$$L(f(x)) = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix}.$$ 

If $n = 2$, is $L$ an isomorphism? What if $n = 3$?
Method: The following diagram outlines a method for determining whether a given linear transformation $T: V \to W$ is an isomorphism, where $V$ and $W$ are finite-dimensional vector spaces.