UNOBSTRUCTED MODULAR DEFORMATION PROBLEMS

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Abstract. Let f be a newform of weight $k \geq 3$ with Fourier coefficients in a number field K. We show that the universal deformation ring of the mod λ Galois representation associated to f is unobstructed, and thus isomorphic to a power series ring in three variables over the Witt vectors, for all but finitely many primes λ of K. We give an explicit bound on such λ for the 6 known cusp forms of level 1, trivial character, and rational Fourier coefficients. We also prove a somewhat weaker result for weight 2.

1. Introduction

Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level N, and character ω; let S be any finite set of places of $\mathbf{Q}$ containing the infinite place and all primes dividing N. For every prime λ of the number field K generated by the a_n, Deligne and Serre have associated to f a semisimple two dimensional Galois representation

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q},S\cup\{\ell\}} \to \mathrm{GL}_2\, k_\lambda$$

over the residue field k_λ of λ; here $G_{\mathbf{Q},S\cup\{\ell\}}$ is the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside S and the characteristic ℓ of k_λ. The representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for almost all λ; for such λ let $R^S_{f,\lambda}$ denote the universal deformation ring parameterizing lifts of $\bar{\rho}_{f,\lambda}$ (up to strict equivalence) to two dimensional representations of $G_{\mathbf{Q},S\cup\{\ell\}}$ over noetherian local rings with residue field k_λ (see Section 2.1 for a precise definition). Using recent work of Diamond, Flach, and Guo [9] on the Bloch–Kato conjectures for adjoint motives of modular forms we prove the following theorem.

Theorem 1. *If $k > 2$, then*

$$R^S_{f,\lambda} \cong W(k_\lambda)[[T_1, T_2, T_3]] \tag{1.1}$$

for all but finitely many primes λ of K (depending on f and S); here $W(k_\lambda)$ is the ring of Witt vectors of k_λ. If $k = 2$, then (1.1) holds for all but finitely many primes λ of K dividing rational primes ℓ such that

$$a_\ell^2 \not\equiv \omega(\ell) \pmod{\lambda}. \tag{1.2}$$

The special case of Theorem 1 for elliptic curves was proven by Mazur [19] using results of Flach [12] on symmetric square Selmer groups of elliptic curves. For weight $k \geq 3$, Theorem 1 answers Mazur's question of [19, Section 11] concerning the finiteness of the set of obstructed primes for modular deformation problems. We refer to [19] for a discussion of additional applications of Theorem 1. Many cases of Theorem 1 have recently been obtained independently by Yamagami [24]; see also [2, 3] for some closely related results which can be used to prove cases of Theorem 1.

Supported by an NSF postdoctoral fellowship.

It is known that the condition (1.2) can fail at most for a set of ℓ of density zero; conjectures of Lang and Trotter predict that it does fail infinitely often, although this is not presently known for any modular forms. See [19, Remark 1 of p. 173] for a discussion.

Our methods are in principle effective: that is, given enough information about the modular form f it is possible to determine a finite set of primes λ containing all those violating (1.1). As a first example, we combine Theorem 1 with work of Hida and Mazur to prove the following theorem.

Theorem 2. *Let f be one of the normalized cusp forms of level 1, weight $k = 12, 16, 18, 20, 22$ or 26, and trivial character. Let $\ell > k+1$ be a prime for which $\bar{\rho}_{f,\ell}$ is absolutely irreducible. Then*

$$R^{\emptyset}_{f,\ell} \cong \mathbf{Z}_\ell[[T_1, T_2, T_3]]. \tag{1.3}$$

For example, for $f = \Delta$, (1.3) holds for $\ell \geq 17$, $\ell \neq 691$.

By [17], to prove Theorem 1 it suffices to show that the Galois cohomology group $H^2(G_{\mathbf{Q}}, \operatorname{ad}\bar{\rho}_{f,\lambda})$ of the adjoint representation of $\bar{\rho}_{f,\lambda}$ vanishes for almost all λ (or for almost all λ satisfying (1.2) when $k = 2$). In Section 2 we explain how to use Poitou–Tate duality and results on Selmer groups as in [9] to reduce the proof of Theorem 1 to two statements:

(1) For fixed p, $H^0(\operatorname{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p), \bar{\varepsilon}_\lambda \otimes \operatorname{ad}\bar{\rho}_{f,\lambda}) = 0$ for almost all λ;
(2) $H^0(\operatorname{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell), \bar{\varepsilon}_\lambda \otimes \operatorname{ad}\bar{\rho}_{f,\lambda}) = 0$ for almost all λ (or for almost all λ satisfying (1.2) when $k = 2$), where $\ell = \operatorname{char} k_\lambda$.

Here $\bar{\varepsilon}_\lambda$ is the mod λ cyclotomic character.

In the elliptic curve case, the restriction of $\bar{\rho}_{f,\ell}$ to the absolute Galois group G_p of $\mathbf{Q}_p$ has a canonical interpretation as the Galois action on the ℓ-torsion points of the elliptic curve; the proofs of (1) and (2) are then a straightforward exercise using formal groups and the Kodaira–Néron classification of special fibers of Néron models of elliptic curves. Unfortunately, for more general modular forms there is no canonical model of the (often reducible) representation $\bar{\rho}_{f,\lambda}|_{G_p}$, and the tools from the elliptic curve case are no longer applicable.

Our proof of (1) divides into two cases. Let π_p be the p-component of the automorphic representation corresponding to f. If π_p is principal series or supercuspidal, then there is no harm in studying the semisimplification of $\bar{\rho}_{f,\lambda}|_{G_p}$. One can then use the local Langlands correspondence for GL_2 to deduce (1) for such p. This is done in Section 3. If π_p is special, then to prove (1) for p it is not sufficient to consider the semisimplification, and the lack of a canonical choice for $\bar{\rho}_{f,\lambda}|_{G_p}$ prevents one from using a purely local approach. Instead, we use a level-lowering argument suggested to us by Ken Ribet to verify (1) for such p. We present this argument in Section 5.2.

We use the theory of Fontaine–Laffaille to reduce (2) to a computation with filtered Dieudonné modules; this is presented in Section 4. The proofs of Theorems 1 and 2 are given in Section 5.

It is a pleasure to thank Matthias Flach, Elena Mantovan, Robert Pollack, and Ken Ribet for helpful conversations related to this work.

Notation. If $\rho : G \to \mathrm{GL}_2 K$ is a representation of a group G over a field K, we write $\operatorname{ad}\rho : G \to \mathrm{GL}_4 K$ for the adjoint representation of G on $\operatorname{End}(\rho)$ and $\operatorname{ad}^0\rho : G \to \mathrm{GL}_3 K$ for the kernel of the trace map from $\operatorname{ad}\rho$ to the trivial representation.

We write $G_{\mathbf{Q}} := \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ for the absolute Galois group of $\mathbf{Q}$. We fix now and forever embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ for each p; these yield injections $G_p \hookrightarrow G_{\mathbf{Q}}$ with $G_p := \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ the absolute Galois group of $\mathbf{Q}_p$. All Frobenius elements are geometric, although for a character ω unramified at ℓ we use the usual normalization $\omega(\ell) := \omega(\mathrm{Frob}_\ell^{-1})$. We use the phrase "almost all" as a synonym for the phrase "all but finitely many".

2. Deformation theory

2.1. **Universal deformation rings.** Let k be a finite field of odd characteristic ℓ. Let S be a finite set of places of $\mathbf{Q}$ containing ℓ and the infinite place, and let $\mathbf{Q}_S$ denote the maximal extension of $\mathbf{Q}$ unramified away from S; set $G_{\mathbf{Q},S} := \mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q})$. Consider an absolutely irreducible continuous Galois representation

$$\bar{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2\, k.$$

We further assume that $\bar{\rho}$ is *odd* in the sense that the image of complex conjugation under $\bar{\rho}$ has distinct eigenvalues.

Let $\mathcal{C}$ denote the category of inverse limits of artinian local rings with residue field k; morphisms in $\mathcal{C}$ are assumed to induce the identity map on k. For a ring A in $\mathcal{C}$, we say that $\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2\, A$ is a *lifting* of $\bar{\rho}$ to A if the composition

$$G_{\mathbf{Q},S} \xrightarrow{\rho} \mathrm{GL}_2\, A \to \mathrm{GL}_2\, k$$

equals $\bar{\rho}$. We say that two liftings ρ_1, ρ_2 of $\bar{\rho}$ to A are *strictly conjugate* if there is a matrix M in the kernel of $\mathrm{GL}_2\, A \to \mathrm{GL}_2\, k$ such that $\rho_1 = M \cdot \rho_2 \cdot M^{-1}$. A *deformation* of $\bar{\rho}$ to A is a strict conjugacy class of liftings of $\bar{\rho}$ to A. The *deformation functor*

$$D_{\bar{\rho}}^S : \mathcal{C} \to \mathrm{Sets}$$

sends a ring A to the set of deformations of $\bar{\rho}$ to A. Since $\bar{\rho}$ is absolutely irreducible, by [17, Proposition 1] the functor $D_{\bar{\rho}}^S$ is represented by a ring $R_{\bar{\rho}}$ of $\mathcal{C}$; that is, there is an isomorphism of functors

$$D_{\bar{\rho}}^S(-) \cong \mathrm{Hom}_{\mathcal{C}}(R_{\bar{\rho}}, -).$$

We call $R_{\bar{\rho}}$ the *universal deformation ring* for the deformation problem $D_{\bar{\rho}}^S$.

The deformation problem $D_{\bar{\rho}}^S$ is said to be *unobstructed* if $H^2(G_{\mathbf{Q},S}, \mathrm{ad}\,\bar{\rho}) = 0$. In this case, [17, Sections 1.7 and 1.10] gives the following description of $R_{\bar{\rho}}$.

Proposition 2.1. *If $D_{\bar{\rho}}^S$ is unobstructed, then*

$$R_{\bar{\rho}} \cong W(k)[[T_1, T_2, T_3]]$$

with $W(k)$ the ring of Witt vectors of k.

2.2. **A criterion for unobstructedness.** We continue with the notation of the previous section. In this section we recall Flach's criterion of [12, Section 3] for the vanishing of $H^2(G_{\mathbf{Q},S}, \mathrm{ad}\,\bar{\rho})$. Let $\mathcal{O} \in \mathcal{C}$ be a totally ramified integral extension of $W(k)$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}$ and fix a lifting

$$\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2\, \mathcal{O}$$

of $\bar{\rho}$ to $\mathcal{O}$. Let K denote the fraction field of $\mathcal{O}$ and let V_ρ (resp. A_ρ) denote a three dimensional K-vector space (resp. $(K/\mathcal{O})^3$) endowed with a $G_{\mathbf{Q},S}$ action via $\mathrm{ad}^0\rho$; we write $V_\rho(1)$ and $A_\rho(1)$ for their Tate twists.

We recall that the *Selmer group* of any these Galois modules M is defined by

$$H^1_f(G_{\mathbf{Q}}, M) := \{c \in H^1(G_{\mathbf{Q}}, M)\,;\, c|_{G_p} \in H^1_f(G_p, M) \text{ for all } p\}$$

where $H^1_f(G_p, M)$ is as in [1, Section 3]. Let V, A denote either of the pairs V_ρ, A_ρ or $V_\rho(1)$, $A_\rho(1)$. We define $Ш(A)$ via the exact sequence

$$0 \to i_* H^1_f(G_{\mathbf{Q}}, V) \to H^1_f(G_{\mathbf{Q}}, A) \to Ш(A) \to 0 \tag{2.1}$$

with $i : V \twoheadrightarrow A$ the natural map. Finally, for any $G_{\mathbf{Q},S}$-module M we define

$$Ш^1(G_{\mathbf{Q},S}, M) := \{c \in H^1(G_{\mathbf{Q},S}, M)\,;\, c|_{G_p} = 0 \text{ for all } p \in S\}.$$

Let $\bar{\varepsilon}_\ell : G_{\mathbf{Q}} \to k^\times$ denote the mod ℓ cyclotomic character.

Proposition 2.2. *Let $\bar{\rho}$ be as above. Suppose that*

1. $H^0(G_p, \bar{\varepsilon}_\ell \otimes \operatorname{ad}\bar{\rho}) = 0$ *for all* $p \in S - \{\infty\}$*;*
2. $H^1_f(G_{\mathbf{Q}}, V_\rho(1)) = 0$*;*
3. $H^1_f(G_{\mathbf{Q}}, A_\rho) = 0$*.*

Then the deformation problem $D^S_{\bar{\rho}}$ is unobstructed.

Proof. The trace pairing on $\operatorname{ad}\bar{\rho}$ identifies $\bar{\varepsilon}_\ell \otimes \operatorname{ad}\bar{\rho}$ with the Cartier dual of $\operatorname{ad}\bar{\rho}$. Poitou–Tate duality (see [20, Proposition 4.10]) thus yields an exact sequence

$$\prod_{p \in S - \{\infty\}} H^0(G_p, \bar{\varepsilon}_\ell \otimes \operatorname{ad}\bar{\rho}) \to \operatorname{Hom}_k\big(H^2(G_{\mathbf{Q},S}, \operatorname{ad}\bar{\rho}), k\big) \to Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}\bar{\rho}). \tag{2.2}$$

The latter group decomposes as

$$Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}\bar{\rho}) = Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell) \oplus Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}^0\bar{\rho}),$$

and the first summand vanishes by [23, Lemma 10.6]. Thus, by (2.2) and hypothesis (1), to prove the proposition it suffices to show that $Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}^0\bar{\rho}) = 0$.

Since ρ is a lifting of $\bar{\rho}$, the $G_{\mathbf{Q},S}$-module $A_\rho(1)[\mathfrak{m}]$ is a realization of $\bar{\varepsilon}_\ell \otimes \operatorname{ad}^0\bar{\rho}$. Since $\bar{\rho}$ is irreducible, the natural map

$$H^1(G_{\mathbf{Q}}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}^0\bar{\rho}) = H^1(G_{\mathbf{Q}}, A_\rho(1)[\mathfrak{m}]) \to H^1(G_{\mathbf{Q}}, A_\rho(1))$$

is injective. The image of $Ш^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \operatorname{ad}^0\bar{\rho})$ under this map is easily seen to lie in $H^1_f(G_{\mathbf{Q}}, A_\rho(1))$, so that to complete the proof it suffices to show that this Selmer group vanishes.

By (2.1) and hypothesis (3) we have $Ш(A_\rho) = 0$. Since $A_\rho(1)$ is Cartier dual to A_ρ, by [11, Theorem 1] this implies that $Ш(A_\rho(1)) = 0$. By (2.1) and hypothesis (2), we conclude that $H^1_f(G_{\mathbf{Q}}, A_\rho(1)) = 0$, as desired. □

3. Local invariants, $\ell \neq p$

3.1. **Characters.** Fix a prime p and let $\mathcal{O} \subseteq \mathbf{C}$ denote the ring of integers of a number field K. For each prime λ of $\mathcal{O}$ not dividing p, fix an isomorphism $\iota_\lambda : \mathbf{C} \xrightarrow{\simeq} \bar{K}_\lambda$ extending the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_\lambda$.

Let F be a finite extension of $\mathbf{Q}_p$; set $G_F = \operatorname{Gal}(\bar{F}/F)$. We say that a continuous character $\chi : F^\times \to \mathbf{C}^\times$ is of *Galois-type* with respect to ι_λ if the character $\iota_\lambda \circ \chi : F^\times \to \bar{K}_\lambda^\times$ extends to a continuous character

$$\chi_\lambda : G_F \to \bar{K}_\lambda^\times$$

via the dense injection $F^\times \hookrightarrow G_F^{\mathrm{ab}}$ of local class field theory. We then write $\bar{\chi}_\lambda : G_F \to \bar{\mathbf{F}}_\ell^\times$ for the reduction of χ_λ. We say that χ is *arithmetic* if $\chi(F^\times) \subseteq \bar{\mathbf{Q}}^\times$.

Lemma 3.1. *Let $\chi : F^\times \to \mathbf{C}^\times$ be an arithmetic character of Galois-type with respect to ι_λ for all $\lambda \nmid p$. If $\bar{\chi}_\lambda = 1$ for infinitely many λ, then $\chi = 1$.*

Proof. Let $\bar{\lambda}$ be the prime of $\bar{\mathbf{Q}}$ which is the kernel of the composition

$$\mathcal{O}_{\bar{\mathbf{Q}}} \xrightarrow{\iota_\lambda} \mathcal{O}_{\bar{K}_\lambda} \to \bar{\mathbf{F}}_\ell.$$

If $\bar{\chi}_\lambda = 1$, then $\chi(F^\times) \subseteq 1 + \bar{\lambda}$. If this holds for infinitely many λ, then $\chi(F^\times) - 1$ lies in primes of $\bar{\mathbf{Q}}$ of infinitely many distinct residue characteristics and thus is trivial, as claimed. □

3.2. **Computation of local invariants.** Let π be an irreducible admissible complex representation of $\mathrm{GL}_2\,\mathbf{Q}_p$. (See [10, Section 11] and the reference therein for definitions, or see [4] for a thoroughly enjoyable introduction.) We say that π is *arithmetic* if either:

(1) π is a subquotient of an induced representation $\pi(\chi_1, \chi_2)$ with each $\chi_i : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ arithmetic; or
(2) π is the base change of an arithmetic character $\chi : F^\times \to \mathbf{C}^\times$ for a quadratic extension $F/\mathbf{Q}_p$; or
(3) π is extraordinary.

We will recall the notion of *Langlands correspondence* (with respect to ι_λ) between π and a λ-adic representation $\rho : G_p \to \mathrm{GL}_2\,\bar{K}_\lambda$ in the course of the proof below. We write $|\cdot| : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ for the norm character; it is arithmetic and of Galois-type with respect to each ι_λ, with $|\cdot|_\lambda$ equal to the λ-adic cyclotomic character $\varepsilon_\lambda : G_p \to K_\lambda^\times$.

Proposition 3.2. *Let π be an arithmetic irreducible admissible complex representation of $\mathrm{GL}_2\,\mathbf{Q}_p$. Let $\{\rho_\lambda : G_p \to \mathrm{GL}_2\,\bar{K}_\lambda\}_{\lambda\nmid p}$ be a family of continuous representations such that π and ρ_λ are in Langlands correspondence with respect to ι_λ for all $\lambda \nmid p$. If π is principal series or supercuspidal, then*

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \mathrm{ad}\,\bar{\rho}_\lambda^{\mathrm{ss}}) = 0$$

for almost all λ.

Proof. For λ of odd residue characteristic we have

$$\bar{\varepsilon}_\lambda \otimes \mathrm{ad}\,\bar{\rho}_\lambda^{\mathrm{ss}} \cong \bar{\varepsilon}_\lambda \oplus \big(\bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0\bar{\rho}_\lambda^{\mathrm{ss}}\big).$$

The first summand has trivial G_p-invariants if and only if λ divides $p-1$, so that we may restrict our attention to $\bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0\bar{\rho}_\lambda^{\mathrm{ss}}$.

Assume first that $\pi = \pi(\chi_1, \chi_2)$ is arithmetic principal series. In this case, for π to be in Langlands correspondence with ρ_λ means that the $\chi_i : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ are of Galois-type with respect to ι_λ and $\rho_\lambda \cong \chi_{1,\lambda} \oplus \chi_{2,\lambda}$. Thus

$$\bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0\bar{\rho}_\lambda^{\mathrm{ss}} \cong \bar{\varepsilon}_\lambda \oplus \bar{\varepsilon}_\lambda\bar{\chi}_{1,\lambda}\bar{\chi}_{2,\lambda}^{-1} \oplus \bar{\varepsilon}_\lambda\bar{\chi}_{1,\lambda}^{-1}\bar{\chi}_{2,\lambda}. \tag{3.1}$$

We must show that each of these characters is non-trivial for almost all λ. As above, this is clear for $\bar{\varepsilon}_\lambda$. If $\bar{\varepsilon}_\lambda\bar{\chi}_{1,\lambda}\bar{\chi}_{2,\lambda}^{-1}$ is trivial for infinitely many λ, then by Lemma 3.1 we must have $\chi_1\chi_2^{-1} = |\cdot|^{-1}$. However, π would then be special or one dimensional, rather than principal series; thus this can not occur. The same argument deals with the characters $\bar{\varepsilon}_\lambda\bar{\chi}_{1,\lambda}^{-1}\bar{\chi}_{2,\lambda}$, settling this case of the proposition.

Next assume that π is supercuspidal but not extraordinary. Then π is the base change of an arithmetic character $\chi : F^\times \to \mathbf{C}^\times$ for a quadratic extension F of $\mathbf{Q}_p$. The Langlands correspondence in this case implies that χ is of Galois-type with respect to each ι_λ, and

$$\rho_\lambda \cong \operatorname{Ind}_{G_F}^{G_p} \chi_\lambda.$$

Let χ^c be the $\operatorname{Gal}(F/\mathbf{Q}_p)$-conjugate character of χ and let $\omega : F^\times \to \mathbf{C}^\times$ denote the character $\chi \cdot (\chi^c)^{-1}$. We have

$$\bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}_\lambda^{\mathrm{ss}} \cong \bar{\varepsilon}_\lambda \chi_F \oplus \left(\bar{\varepsilon}_\lambda \otimes \operatorname{Ind}_{G_F}^{G_p} \bar{\omega}_\lambda\right)$$

with $\chi_F : \operatorname{Gal}(F/\mathbf{Q}_p) \to \{\pm 1\}$ the non-trivial character for $F/\mathbf{Q}_p$. The first summand is not a problem for $\ell > 3$. The second summand is irreducible if and only if $\bar{\omega}_\lambda \neq \bar{\omega}_\lambda^{\mathrm{c}}$. In particular, if $\omega \neq \omega^{\mathrm{c}}$, then this case of the proposition follows from Lemma 3.1. If instead $\omega = \omega^{\mathrm{c}}$, then $\omega^2 = 1$, ω extends to a character $\tilde{\omega} : \mathbf{Q}_p^\times \to \mathbf{C}^\times$, and

$$\bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}_\lambda^{\mathrm{ss}} \cong \bar{\varepsilon}_\lambda \chi_F \oplus \bar{\varepsilon}_\lambda \bar{\tilde{\omega}}_\lambda \oplus \bar{\varepsilon}_\lambda \bar{\tilde{\omega}}_\lambda^{-1}.$$

Since $\omega^2 = 1$, we clearly have $\tilde{\omega} \neq |\cdot|^{\pm 1}$; this case of the proposition thus again follows from Lemma 3.1.

Finally, if π is extraordinary, then $p = 2$, the image $\operatorname{proj} \rho_\lambda(I_2)$ of inertia in $\operatorname{PGL}_2 \bar{\mathbf{Q}}_\ell$ is isomorphic to A_4 or S_4, and the composition

$$\operatorname{proj} \rho_\lambda(I_2) \hookrightarrow \operatorname{PGL}_2 \bar{\mathbf{Q}}_\ell \xrightarrow{\operatorname{ad}^0} \operatorname{GL}_3 \bar{\mathbf{Q}}_\ell$$

is an irreducible representation of $\operatorname{proj} \rho_\lambda(I_2)$. Since $\operatorname{proj} \rho_\lambda(I_2)$ has order 12 or 24, it follows that $\operatorname{ad}^0 \bar{\rho}_\lambda = \operatorname{ad}^0 \bar{\rho}_\lambda^{\mathrm{ss}}$ is an irreducible $\bar{\mathbf{F}}_\ell$-representation of I_2 for λ of residue characteristic at least 5. Thus already $H^0(I_2, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}_\lambda^{\mathrm{ss}}) = 0$ for such λ; the proposition follows. □

Remark 3.3. Note that if $\bar{\rho} : G_p \to \operatorname{GL}_2 \bar{\mathbf{F}}_\ell$ is any reduction of ρ, then

$$\dim_{\bar{\mathbf{F}}_\ell} H^0(G_p, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}) \leq \dim_{\bar{\mathbf{F}}_\ell} H^0(G_p, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}^{\mathrm{ss}}).$$

Remark 3.4. Suppose that $\pi = \pi(\chi_1, \chi_2)$ is an unramified principal series representation (that is, $\chi_i(\mathbf{Z}_p^\times) = 1$ for $i = 1, 2$). Then if π is in Langlands correspondence with ρ_λ, using (3.1) one finds that:

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}_\lambda^{\mathrm{ss}}) \neq 0 \Rightarrow$$
$$p\big(\chi_1(p) + \chi_2(p)\big)^2 \equiv (p+1)^2 \chi_1(p) \chi_2(p) \pmod{\lambda}.$$

In the case of a representation $\rho_{f,\lambda}$ associated to a newform $f = \sum a_n q^n$ of weight k, level N, and character ω (see Section 5.1), this translates to the condition:

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}_{f,\lambda}^{\mathrm{ss}}) \neq 0 \;\Rightarrow\; a_p^2 \equiv p^{k-2}(p+1)^2 \omega(p) \pmod{\lambda}$$

for $p \nmid N\ell$.

Remark 3.5. If π is one dimensional or special, then

$$\dim_{\bar{\mathbf{F}}_\ell} H^0(G_p, \bar{\varepsilon}_\lambda \otimes \operatorname{ad}^0 \bar{\rho}^{\mathrm{ss}}) = 1$$

for almost all λ. For π special, however, one might hope to obtain an analogue of Proposition 3.2 by allowing $\bar{\rho}$ to be non-semisimple. Unfortunately, there is no way to formulate such a result purely locally. Instead, for the case of special local components of Galois representations attached to modular forms, in Section 5.2

we will use global considerations to rigidify $\bar{\rho}$; we will then be able to obtain the required vanishing in the special case as well.

4. Local invariants, $\ell = p$

4.1. **Filtered Dieudonné modules.** We briefly review the theory of Fontaine–Laffaille [13]. Let K be a finite extension of $\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $v : K^\times \to \mathbf{Q}$ be the valuation on K, normalized so that $v(\ell) = 1$; let λ be a uniformizer of $\mathcal{O}$ and let $e = v(\lambda)^{-1}$ denote the absolute ramification degree of K.

Definition 4.1. A *filtered Dieudonné $\mathcal{O}$-module (over $\mathbf{Z}_\ell$)* is an $\mathcal{O}$-module D of finite type, endowed with a decreasing filtration $(D^i)_{i\in\mathbf{Z}}$ by $\mathcal{O}$-module direct summands and a family $(f_i : D^i \to D)_{i\in\mathbf{Z}}$ of $\mathcal{O}$-linear maps satisfying:

(1) $D^i = D$ (resp. $D^i = 0$) for $i \ll 0$ (resp. $i \gg 0$);
(2) $f_i|_{D^{i+1}} = \ell \cdot f_{i+1}$ for all i;
(3) $D = \sum_{i\in\mathbf{Z}} f_i(D^i)$.

For $a < b$, let $\mathcal{MF}^{a,b}(\mathcal{O})$ denote the category of filtered Dieudonné $\mathcal{O}$-modules D satisfying $D^a = D$ and $D^b = 0$.

We now recall the relation between filtered Dieudonné $\mathcal{O}$-modules and Galois representations. Let V be a finite dimensional K-vector space with a continuous K-linear action of G_ℓ. Define a K-vector space

$$D_{\mathrm{crys}}(V) := (B_{\mathrm{crys}} \otimes_{\mathbf{Q}_\ell} V)^{G_\ell}$$

with B_{crys} the crystalline period ring of Fontaine; $D_{\mathrm{crys}}(V)$ inherits a decreasing filtration $(D^i_{\mathrm{crys}}(V))_{i\in\mathbf{Z}}$ from the filtration on B_{crys}. We say that V is *crystalline* if $D_{\mathrm{crys}}(V)$ and V are K-vector spaces of the same dimension.

For $a < b$, let $\mathcal{G}^{a,b}(\mathcal{O})$ denote the category of finite type $\mathcal{O}$-module subquotients of crystalline K-representations V with $D^a_{\mathrm{crys}}(V) = D_{\mathrm{crys}}(V)$ and $D^b_{\mathrm{crys}}(V) = 0$. Fontaine–Laffaille define a functor

$$\mathcal{U} : \mathcal{MF}^{a,a+\ell}(\mathcal{O}) \to \mathcal{G}^{a,a+\ell}(\mathcal{O})$$

which is equivalent to the identity functor on the underlying $\mathcal{O}$-modules and which induces an equivalence of categories between $\mathcal{MF}^{a,a+\ell-1}(\mathcal{O})$ and $\mathcal{G}^{a,a+\ell-1}(\mathcal{O})$. (See [9, Section 1.1] for more details; note that we are using Tate twists as in [1, Section 4] to extend $\mathcal{U}$ to the case $a \neq 0$.)

Example 4.2. Let $\omega : G_\ell \to \mathcal{O}^\times$ be an unramified character of finite order and let $\mathcal{O}(\omega)$ denote a free $\mathcal{O}$-module of rank 1 with G_ℓ-action via ω. Then $\mathcal{O}(\omega) \in \mathcal{G}^{0,1}(\mathcal{O})$, so that there is $D_\omega \in \mathcal{MF}^{0,1}(\mathcal{O})$ such that $\mathcal{U}(D_\omega) \cong \mathcal{O}(\omega)$. This D_ω is a free $\mathcal{O}$-module of rank one with $D_\omega = D^0_\omega$ and f_0 multiplication by $\omega^{-1}(\ell)$.

Example 4.3. Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level N, and character ω. Assuming that K contains some completion of the number field generated by the a_n, there is a Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2 K$ associated to f as in Section 5.1. Fix an embedding $G_\ell \hookrightarrow G_{\mathbf{Q}}$ and let V_f be a two dimensional K-vector space on which G_ℓ acts via $\rho_f|_{G_\ell}$.

Fix a G_ℓ-stable $\mathcal{O}$-lattice $T_f \subseteq V_f$. If $\ell \nmid N$, then V_f is crystalline and $T_f \in \mathcal{G}^{0,k}(\mathcal{O})$. Thus for $\ell > k$ there exists $D_f \in \mathcal{MF}^{0,k}(\mathcal{O})$ with $\mathcal{U}(D_f) \cong T_f$. Using [1, Theorem 4.3] and standard properties of modular representations, one obtains the following description of D_f. The filtration satisfies:

$$\operatorname{rank}_{\mathcal{O}} D_f^i = \begin{cases} 2 & i \le 0; \\ 1 & 1 \le i \le k-1; \\ 0 & k \le i. \end{cases}$$

Choose an $\mathcal{O}$-basis x, y of D_f with x an $\mathcal{O}$-generator of D_f^1. Let $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ be such that

$$f_0 x = \alpha x + \beta y; \qquad f_0 y = \gamma x + \delta y.$$

Then $\alpha + \delta = a_\ell$ and $\alpha\delta - \beta\gamma = \ell^{k-1}\omega(\ell)$. By (2) of Definition 4.1 we have $v(\alpha), v(\beta) \ge k-1$.

4.2. **Computation of local invariants.** Let f and T_f be as in Example 4.3. Let

$$\bar{\rho}_f : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\lambda)$$

be the Galois representation on $T_f/\lambda T_f$.

Proposition 4.4. *Assume $\ell \nmid N$ and $\ell > 2k$. Then*

$$H^0(G_\ell, \bar{\varepsilon}_\lambda \otimes \operatorname{ad} \bar{\rho}_f) = 0$$

unless $k = 2$ and $a_\ell^2 \equiv \omega(\ell) \pmod{\lambda}$.

Proof. Since $\det \rho_f = \varepsilon_\lambda^{1-k}\omega^{-1}$, we have

$$\operatorname{ad} T_f(1) \cong \big(T_f \otimes_{\mathcal{O}} T_f \otimes_{\mathcal{O}} \mathcal{O}(\omega)\big)(k) \tag{4.1}$$

(where $(-)(k)$ denotes the k-fold Tate twist). It follows that $\operatorname{ad} T_f(1) \in \mathcal{G}^{-k,k-1}(\mathcal{O})$. Since $\ell > 2k$, there thus exists $D \in \mathcal{MF}^{-k,k-1}(\mathcal{O})$ with $\mathcal{U}(D) \cong \operatorname{ad} T_f(1)$. In fact, by (4.1) and [14, Proposition 1.7] we can take

$$D = \big(D_f \otimes_{\mathcal{O}} D_f \otimes_{\mathcal{O}} D_\omega\big)(k)$$

with notation as in Examples 4.2 and 4.3. Further, since $\operatorname{ad} T_f(1)/\lambda$ is a realization of $\bar{\varepsilon}_\lambda \otimes \operatorname{ad} \bar{\rho}_f$, by [1, Lemma 4.5] we have

$$H^0(G_\ell, \bar{\varepsilon}_\lambda \otimes \operatorname{ad} \bar{\rho}_f) \cong \ker(1 - f_0 : D^0/\lambda D^0 \to D/\lambda D). \tag{4.2}$$

Thus to prove the proposition it suffices to compute the latter group.

By the definition of Tate twists and tensor products of filtered Dieudonné $\mathcal{O}$-modules, we have

$$\begin{aligned} D^0 &= (D_f \otimes D_f \otimes D_\omega)^k \\ &= \sum_{i_1+i_2+i_3=k} D_f^{i_1} \otimes D_f^{i_2} \otimes D_\omega^{i_3} \\ &= D_f^1 \otimes D_f^{k-1} \otimes D_\omega^0 \\ &= \mathcal{O} \cdot (x \otimes x \otimes w) \end{aligned}$$

where x is as in Example 4.3 and w is an $\mathcal{O}$-generator of D_ω. Using (2) of Definition 4.1, we compute:

$$\begin{aligned} f_0(x \otimes x \otimes w) &:= f_1 x \otimes f_{k-1} x \otimes f_0 w \\ &= \frac{\omega^{-1}(\ell)}{\ell^k}(\alpha x + \beta y) \otimes (\alpha x + \beta y) \otimes w. \end{aligned} \tag{4.3}$$

Suppose now that (4.2) is non-zero. Then by (4.3) we must have

$$\omega^{-1}(\ell)\alpha^2 \equiv \ell^k \pmod{\lambda^{ek+1}}; \tag{4.4}$$

$$\alpha\beta \equiv 0 \pmod{\lambda^{ek+1}}. \tag{4.5}$$

In particular, we must have $v(\alpha) = k/2$. Since also $v(\alpha) \geq k-1$, this implies that $k = 2$ and $v(\alpha) = 1$. Thus (4.5) implies that $v(\beta) > 1$. As $\alpha\delta - \beta\gamma = \ell\omega(\ell)$, we conclude that $v(\delta) = 0$ and

$$\alpha\delta \equiv \ell\omega(\ell) \pmod{\lambda^{e+1}}.$$

Using this and (4.4) one deduces easily that

$$a_\ell^2 = (\alpha + \delta)^2 \equiv \omega(\ell) \pmod{\lambda}$$

as claimed. □

5. Unobstructed deformation problems

5.1. **Modular representations.** Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level N, and character ω. Fix a finite set of primes S containing all primes dividing N. Let K be the number field generated by the a_n. Then for any prime λ of K (say with residue field k_λ of characteristic ℓ) there is a continuous Galois representation

$$\rho_{f,\lambda} : G_{\mathbf{Q},S\cup\{\ell\}} \to \mathrm{GL}_2 K_\lambda,$$

unramified with $\operatorname{tr}\rho_{f,\lambda}(\mathrm{Frob}_p) = a_p$ for $p \nmid N\ell$; the determinant of $\rho_{f,\lambda}$ is $\varepsilon_\lambda^{1-k}\omega^{-1}$. (We find it more convenient to work with this geometric normalization of $\rho_{f,\lambda}$, which is dual to the more common arithmetic normalization.) Let

$$\bar\rho_{f,\lambda} : G_{\mathbf{Q},S\cup\{\ell\}} \to \mathrm{GL}_2 k_\lambda.$$

be the semisimple reduction of $\rho_{f,\lambda}$. Then $\bar\rho_{f,\lambda}$ is absolutely irreducible for almost all primes λ by [9, Lemma 7.13].

Let π be the automorphic representation corresponding to f as in [10, Section 11.1] and let $\pi = \otimes' \pi_p$ be the decomposition of π into admissible complex representations π_p of $\mathrm{GL}_2 \mathbf{Q}_p$. Fix isomorphisms $\iota_\lambda : \mathbf{C} \xrightarrow{\simeq} \bar K_\lambda$ (extending the inclusion $K \hookrightarrow K_\lambda$) for all λ. By [6, Théorème B] and [16, Proposition 9.3] each π_p is arithmetic and infinite dimensional and is in Langlands correspondence (with respect to ι_λ) with $\rho_{f,\lambda}|_{G_p}$ for each $\lambda \nmid p$.

5.2. **Special primes.** We continue with the notation of the previous section. Let p be a prime such that π_p is special; that is, π is the unique infinite dimensional subquotient of $\pi(\chi\,|\cdot|\,,\chi)$ for an arithmetic character $\chi : \mathbf{Q}_p^\times \to \mathbf{C}^\times$. Then χ is of Galois-type with respect to ι_λ for each $\lambda \nmid p$, and

$$\rho_{f,\lambda}|_{G_p} \cong \begin{pmatrix} \varepsilon_\lambda\chi_\lambda & * \\ 0 & \chi_\lambda \end{pmatrix}$$

with the upper right-hand corner non-zero and ramified.

Lemma 5.1. *If* $p^2 \not\equiv 1 \pmod{\lambda}$, *then*

$$\bar\rho_{f,\lambda}|_{G_p} \otimes \bar k_\lambda \cong \begin{pmatrix} \bar\varepsilon_\lambda\bar\chi_\lambda & * \\ 0 & \bar\chi_\lambda \end{pmatrix}.$$

Proof. Since the semisimplification of $\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda$ is $\bar{\varepsilon}_\lambda \bar{\chi}_\lambda \oplus \bar{\chi}_\lambda$, the only way the lemma can fail is if

$$\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda \cong \begin{pmatrix} \bar{\chi}_\lambda & \nu \\ 0 & \bar{\varepsilon}_\lambda \bar{\chi}_\lambda \end{pmatrix}$$

with ν non-trivial. One checks directly that $\bar{\varepsilon}_\lambda^{-1} \bar{\chi}_\lambda^{-1} \nu$ is naturally an element of $H^1(G_p, \bar{k}_\lambda(-1))$; since this cohomology group is trivial unless $p^2 \equiv 1 \pmod{\lambda}$, the lemma follows. □

Using Lemma 5.1, the proof of the next lemma is a straightforward matrix computation; we omit the details.

Lemma 5.2. *Let p be a prime for which π_p is special. Let λ be a prime not dividing $2p(p^2-1)$. Then*

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0 \bar{\rho}_{f,\lambda}) \neq 0$$

if and only if $\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda$ is semisimple.

The level-lowering approach of the next proposition was suggested to the author by Ken Ribet.

Proposition 5.3. *Let p be a prime for which π_p is special. Then*

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \mathrm{ad}\, \bar{\rho}_{f,\lambda}) = 0$$

for almost all primes λ.

Proof. As always it suffices to prove the proposition for $\bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0 \bar{\rho}_{f,\lambda}$. Let χ be as above. Since the automorphic representation π has central quasi-character $|\cdot|^{1-k}\,\omega^{-1}$, we must have $\chi^2 = |\cdot|^{-k}\,\omega_p^{-1}$, where $\omega_p : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ is the p-component of ω. In particular, $\chi_p' := \chi^{-1}\,|\cdot|^{-k/2}$ has finite order. Extend χ_p' to a Dirichlet character χ' and let f' denote the newform of weight k and some level M associated to the eigenform $f \otimes \chi'$. The p-component π_p' of the automorphic representation associated to f' is a subquotient of $\pi(\chi_p'\chi\,|\cdot|\,, \chi_p'\chi)$; since $\chi_p'\chi$ is unramified at p by construction, we conclude that p divides M exactly once.

Suppose now that $\lambda \nmid 2p(p^2-1)$ is such $\bar{\rho}_{f,\lambda}$ is irreducible and

$$H^0(G_p, \bar{\varepsilon}_\lambda \otimes \mathrm{ad}^0 \bar{\rho}_{f,\lambda}) \neq 0. \tag{5.1}$$

Then $\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda$ is semisimple by Lemma 5.2, so that

$$\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda \cong \bar{\varepsilon}_\lambda \bar{\chi}_\lambda \oplus \bar{\chi}_\lambda$$

by Lemma 5.1. Thus

$$\bar{\rho}_{f',\lambda}|_{G_p} \cong \bar{\rho}_{f,\lambda}|_{G_p} \otimes \chi_p' \cong \bar{\varepsilon}_\lambda^{1-\frac{k}{2}} \oplus \bar{\varepsilon}_\lambda^{-\frac{k}{2}}.$$

In particular $\bar{\rho}_{f',\lambda}$ is unramified at p. By [7, Theorem 1.1] applied to f' there is then a newform f'' of weight k and level dividing M/p with f' congruent to f'' modulo some prime of $\bar{\mathbf{Q}}$ above λ. However, there are only finitely many newforms f'' of weight k and level dividing M/p, and for each such f'' there are only finitely many λ with f' congruent to f'' modulo some prime of $\bar{\mathbf{Q}}$ above λ. Thus (5.1) can only hold for finitely many λ; the proposition follows. □

5.3. **Proof of Theorem 1.** Let f and S be as above. For any λ at which $\bar{\rho}_{f,\lambda}$ is absolutely irreducible, we write $R^S_{f,\lambda}$ for the universal deformation ring for the deformation problem $D^{S\cup\{\ell\}}_{\bar{\rho}_{f,\lambda}}$ as in Section 2.1.

Theorem 5.4. *Suppose $k > 2$. Then*

$$R^S_{f,\lambda} \cong W(k_\lambda)[[T_1, T_2, T_3]]$$

for almost all primes λ.

Proof. We apply Proposition 2.2 to the lifting $\rho_{f,\lambda}$ of $\bar{\rho}_{f,\lambda}$. Hypothesis (1) holds for almost all λ by Proposition 3.2 and Remark 3.3 (for $p \in S$ for which π_p is not special), Proposition 5.3 (for $p \in S$ at which π_p is special), and Proposition 4.4 (for $p = \ell$). Hypotheses (2) and (3) hold for almost all λ by [9, Theorems 8.2 and 7.15] and the fact that (in the notation of [9])

$$H^1_f(G_{\mathbf{Q}}, A_{\rho_{f,\lambda}}) \subseteq H^1_\Sigma(G_{\mathbf{Q}}, A_{\rho_{f,\lambda}})$$

for any finite set of primes Σ. Thus the deformation problem $D^{S\cup\{\ell\}}_{\bar{\rho}_{f,\lambda}}$ is unobstructed for almost all λ; Proposition 2.1 completes the proof. □

Theorem 5.5. *Suppose $k = 2$. Then*

$$R^S_{f,\lambda} \cong W(k_\lambda)[[T_1, T_2, T_3]] \tag{5.2}$$

for almost all primes λ dividing ℓ with $a_\ell^2 \not\equiv \omega(\ell) \pmod{\lambda}$. In particular, (5.2) holds for a set of λ of density one.

Proof. The proof of the first statement is the same as the proof of Theorem 5.4, taking into account the modifications for $k = 2$ in Proposition 4.4. The density statement follows from [22, Theorem 20] and the Ramanujan–Petersson conjecture (proven by Deligne). □

Note that by [17, Section 1.3], the analogue of these results for the arithmetic normalization of $\bar{\rho}_{f,\lambda}$ hold as well.

5.4. **Modular forms of level one.** Let f be the unique normalized cusp form of level 1, weight $k = 12, 16, 18, 20, 22$ or 26, and trivial character. Then f has rational coefficients, so that for every prime ℓ we obtain a representation

$$\bar{\rho}_{f,\ell} : G_{\mathbf{Q},\{\ell\}} \to \mathrm{GL}_2\,\mathbf{F}_\ell.$$

We recall the set of primes ℓ for which $\bar{\rho}_{f,\ell}$ is not absolutely irreducible (see [21]):

k	ℓ	k	ℓ
12	2,3,5,7,691	20	2,3,5,7,11,13,283,617
16	2,3,5,7,11,3617	22	2,3,5,7,13,17,131,593
18	2,3,5,7,11,13,43867	26	2,3,5,7,11,17,19,657931

Theorem 5.6. *Let f be as above and let $\ell > k + 1$ be a prime for which $\bar{\rho}_{f,\ell}$ is absolutely irreducible. Then*

$$R^\emptyset_{f,\ell} \cong \mathbf{Z}_\ell[[T_1, T_2, T_3]]. \tag{5.3}$$

We leave it to the reader to use Remark 3.4 to derive the generalization of Theorem 5.6 to the case $S \neq \emptyset$.

Proof. Fix $\ell > 2k$. Then by [9, Corollary 7.2 and Section 7.4] we have

$$\#H^1_f(G_{\mathbf{Q}}, A_{\rho_{f,\ell}}) = \#H^1_\emptyset(G_{\mathbf{Q}}, A_{\rho_{f,\ell}}) = \#\mathbf{Z}_\ell/\eta_f^\emptyset \mathbf{Z}_\ell \tag{5.4}$$

where $\eta_f^\emptyset \subseteq \mathbf{Z}$ is the congruence ideal of [9, Section 6.4]. By definition, $\eta_f^\emptyset$ is generated by the element $d(f)$ of [15, Theorem 1]. Using [15, Theorem 1 and Theorem 2] and the fact that f is the unique normalized cusp form of its weight, level, and character, we see that $d(f)$ is divisible only by primes $\leq k-2$. In particular, $d(f)$ is not divisible by ℓ, so that (5.4) implies that

$$H^1_f(G_{\mathbf{Q}}, A_{\rho_{f,\ell}}) = 0. \tag{5.5}$$

Since $S = \emptyset$, Propositions 2.2 and 4.4 now imply that (5.3) holds for $\ell > 2k$. As one may check explicitly that each of these forms is ordinary for for $k+1 < \ell < 2k$, (5.3) for such ℓ follows on checking the criterion of [18, Section 7]. (Note that the Conjecture of [18, Section 6] is known to hold by results of Diamond [8]; see [5, proof of Theorem 2] for details.) □

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