# On Selmer Groups of Geometric Galois Representations

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*Dedicated*

*to the memory of Annalee Henderson*

*and to Arnold Ross*

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# Introduction

Fix a squarefree integer $N$ and let $f$ be a rational weight 2 newform for $\Gamma_0(N)$. Let $H$ be the $l$-adic representation associated to $f$ for some $l \geq 7$; $H$ is a free $\mathbf{Z}_l$-module of rank 2. Let $\mathbf{Q}_S$ denote the maximal extension of $\mathbf{Q}$ unramified away from $Nl$ and set $G_{\mathbf{Q}_S} = \mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q})$. Flach proved the following theorem regarding the deformation theory of $H$. (See [**Wes00**, Appendix A] for the proof that the set of primes satisfying Flach's conditions has density 1; this set can be given quite explicitly.)

THEOREM 0.1 ([**Fla92**]). *Fix $f$ as above and assume that $f$ does not have complex multiplication. Then the set of primes $l$ such that the universal deformation ring of the residual representation $\rho : G_{\mathbf{Q}_S} \to \mathrm{Aut}_{\mathbf{F}_l}(H/lH)$ is isomorphic to $\mathbf{Z}_l[[T_1, T_2, T_3]]$ has density* 1.

This result was extended to the case of newforms defined over arbitrary number fields in [**Fla95**] and [**Maz**]. In this case the $l$-adic representation $H$ is free of rank 2 over a certain completion $A$ of a Hecke algebra; this ring $A$ is a reduced, finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra and contains a Hecke operator $T_p$ for every prime $p$. Let $k$ denote the residue field of $A$. Mazur observed that Flach's construction can be used to obtain results on the Taylor-Wiles deformation problem for almost all $l \geq 7$ not dividing $N$.

THEOREM 0.2 ([**Maz**]). *Let $f$ be a newform of weight* 2 *for $\Gamma_0(N)$. Let $l \geq 7$ be a prime not dividing $N$ and let $H$ be the $l$-adic representation associated to $f$. Assume that the natural map $G_{\mathbf{Q}} \to \mathrm{Aut}_k(H \otimes_A k)$ is surjective. Let $R$ be the minimally ramified universal deformation ring for $H \otimes_A k$. Then $R$ is a finite $A$-algebra and the natural map $R \to A$ induces an isomorphism of differentials $\Omega_R \otimes_R A \to \Omega_A$.*

In this thesis I extend the results of Flach and Mazur to the case of (most) newforms of weight at least 2 for $\Gamma_1(N)$. I also show that the "geometric Euler system" used to prove these results has a very rich algebraic structure and that the isomorphism it yields in deformation theory is essentially canonical. The main result in this context can be phrased as follows.

THEOREM 0.3 (Theorem XII.3.1 and Theorem IV.6.2). *Let $f$ be a newform of weight $k \geq 2$ for $\Gamma_1(N)$. Let $l \geq \max\{7, k\}$ be a prime not dividing $N$ and let $H$ be the $l$-adic representation (over an appropriate completion $A$ of a Hecke algebra) associated to $f$. Assume that $f$ can be "cleanly realized" in the cohomology of the universal elliptic curve with level $N$-structure (see Chapter XII for precise conditions). Set $T = \mathrm{End}^0_A H(1)$ and assume that the natural map $G_{\mathbf{Q}} \to \mathrm{Aut}_k(H \otimes_A k)$ is surjective. Let $R$ be the minimally ramified universal deformation ring for $H \otimes_A k$ (see Section IV.1). Then $R$ is a finite $A$-algebra and the natural map $R \to A$ induces an isomorphism of differentials $\Omega_R \otimes_R A \to \Omega_A$. Furthermore, the inverse of* *this isomorphism is characterized by the fact that the identification*

$$\Omega_A \to \Omega_R \otimes_R A \cong \mathrm{Hom}_{\mathbf{Z}_l}\big(H^1_f(\mathbf{Q}, T^*[\eta]), \mathbf{Q}_l/\mathbf{Z}_l\big)$$

*(see Section IV.2) identifies the differential of $T_p \in A$ with $-12$ times the image under the Bockstein pairing*

$$H^1_f(\mathbf{Q}, T/\eta T) \to \mathrm{Hom}_{\mathbf{Z}_l}\big(H^1_f(\mathbf{Q}, T^*[\eta]), \mathbf{Q}_l/\mathbf{Z}_l\big)$$

*of the cohomology class $c_p \in H^1_f(\mathbf{Q}, T/\eta T)$ obtained via the Flach construction. Here $H^1_f(\mathbf{Q}, \cdot)$ is a Selmer group (see Section II.1), $T^*$ is the Cartier dual of $T$, and $\eta$ is the congruence element (see Appendix B.2) for the Gorenstein $\mathbf{Z}_l$-algebra $A$.*

However, it is natural to hope that the method of proof of these results is as interesting as the results themselves. For this reason we proceed in as much generality as we can. Let $X$ be a nonsingular algebraic variety over a global field $F$ and let $H$ be a quotient of the étale cohomology group $H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))$ for some $m$. I give a "general" method (contingent on the existence of appropriate geometric data on $X \times X$) for the production of geometric Euler systems for $\mathrm{End}^0_A H$; these in turn yield corresponding annihilators of certain Selmer groups. These annihilators yield results on the deformation theory of the Galois representation $H/lH$. Somewhat more generally one can hope to use appropriate geometric data on $X$ itself to control the Selmer group of $H$; this could then possibly be related to the Bloch-Kato conjectures.

The required geometric data (in the deformation theory case) does exist for modular curves and Kuga-Sato varieties. It seems likely that it exists in the case of Hilbert modular surfaces as well. One can give "explicit" conditions for the existence of this data in general, although at this point these are not particularly useful.

We now discuss the contents of this thesis in more detail. The first five chapters concern Selmer groups and geometric Euler systems in Galois cohomology. The material of the first three chapters is presented in a fair amount of generality; Chapters IV and V are focused on the specific Taylor-Wiles deformation problem. The material of the first two chapters is essentially standard, although our presentation is a synthesis of several others.

Chapter I concerns local conditions on Galois cohomology. We define such conditions in full generality in preparation for later results. We especially focus on the functorial aspects of these conditions. We also include the explicit computation of the "natural" local condition in the case of ordinary representations.

Chapter II is the globalization of Chapter I: we use local conditions at every place to define Selmer groups of Galois modules. After establishing appropriate functorialities we turn to the definition of two pairings which play a crucial role in our work. The first, the Kolyvagin pairing, combines local and global information. We prove that in a certain sense this pairing evaluates how far a collection of local cohomology classes are from arising from a global cohomology class. The second pairing, the Bockstein pairing, is a pairing between Selmer groups which is of independent interest in many circumstances.

In Chapter III we define the notion of a partial geometric Euler system and prove the corresponding annihilation theorems for Selmer groups via the Kolyvagin pairing and the Tchebatorev density theorem. We also give an application of these methods to prove the non-degeneracy of the Bockstein pairing in the presence of an Euler system. These results are all based on the ideas of Thaine and Kolyvagin, as refined by Flach, Mazur and Rubin. Our presentation is marginally more general than others, but otherwise is well-known.

Chapter IV concerns Mazur's notions of full geometric Euler systems. We begin by considering the deformation theory of certain rank 2 Galois representations over $\mathbf{Z}_l$-algebras $A$ and explain the connection with Selmer groups. We then define the notion of a Flach system, which is a slightly strengthened partial geometric Euler system. The real refinement comes with Mazur's cohesive Flach systems, which combine Flach systems with some additional global algebraic structure. This additional structure results in much more precise deformation theoretic consequences. We then refine this further with our new notion of a cohesive Flach system of Eichler-Shimura type; these are cohesive Flach systems with sharply specified local behavior.

One can pass via the Bockstein pairing and a cohesive Flach system to a certain complicated pairing between the differentials $\Omega_A$ and their dual. In Chapter V we give a computational proof that in the case of a cohesive Flach system of Eichler-Shimura type this pairing is nothing more than a scalar multiple of the canonical duality pairing. This is mostly straightforward except for the computation of the local invariant map and a certain matrix lemma.

The second part of this thesis concerns the production of geometric Euler systems for the étale cohomology of varieties over number fields. The central tool, which we call the Flach map, originated in the work of Flach. Our description is a generalization of that of [**Fla95**] to higher dimensional varieties. [**Maz**] provided an alternate description in the low-dimensional case; this description does not make direct use of algebraic $K$-theory. We have not adopted this approach as it does not seem to easily generalize to higher dimensions.

The Flach map is defined in Chapter VI, after some preliminaries on the coniveau spectral sequence in étale cohomology and algebraic $K$-theory. We define the Flach map for smooth separated schemes $X$ over any perfect field and over some non-perfect fields as well; it is a map from certain pairs of cycles and functions on $X$ to the Galois cohomology of the étale cohomology of $X$. The description of these cycles and functions is most straightforward over local and global fields.

In Chapter VII we give the fundamental local description of the Flach map for a variety $X$ over a global field. Specifically, we prove that at places of good reduction one can test a Galois cohomology class against an appropriate unramified local condition via a certain divisor map to the Chow groups of $X$. (Flach proved a similar result for products of elliptic curves in [**Fla92**]; he offered a result for products of modular curves in [**Fla95**], although the proof there is incomplete.) The proof of this theorem makes use of certain results from higher algebraic $K$-theory; the statement, however, does not involve any explicit $K$-theory. Assuming a detailed knowledge of the geometry of $X$, this makes it possible to use the Flach map to generate partial geometric Euler systems; this is the subject of Chapter IX. We also give a result of Flach concerning the local behavior of the Flach map to $l$-adic cohomology at places above $l$.

Chapter VIII concerns certain algebraic structures on the Flach map for products $X \times X$. After some preliminary discussion of algebraic correspondences, we prove the fundamental Leibniz relation of Mazur and Beilinson. Their argument for this relation for curves used a clever reduction to a trivial case. We instead present a more direct proof which is valid in arbitrary dimension. We then explain how to use the Leibniz relation to generate a system of Galois cohomology classes which have the algebraic structure of a cohesive Flach system. This involves the theory of Gorenstein rings and bilateral derivations as developed in Appendix B.

Chapter IX combines the results of Chapter VII and Chapter VIII to give some sample theorems for the production of geometric Euler systems. In all cases we must assume the existence of appropriate geometric data. The ideas of Section IX.3 were inspired by [?] and [?], although these papers do not explicitly appear anywhere in the discussion.

The last three chapters are the applications of the methods we have developed to modular forms. Chapter X concerns the case of the modular curve $X_0(N)$. We make some auxiliary hypotheses on the Hecke algebra to simplify the exposition; with these in place the construction is straightforward. (The existence of the cohesive Flach system in this case is due to Mazur.) These restrictions are removed via the more general results on $X_1(N)$ in Chapter XI. In Chapter XII we use the realization of Galois representations for modular forms of higher weight in the cohomology of open Kuga-Sato varieties to construct the cohesive Flach system. This requires some slight extensions of the results of Chapters VII and VIII to this setting. Otherwise the construction is completely analogous to the previous cases. The constructions of Chapters XI and XII are new, although in some cases the deformation theoretic conclusions are weaker than those of [?] (for the weight 2 case) and upcoming work of Diamond (for the higher weight case).

We include two appendices. The first concerns compatibilities of edge maps of spectral sequences; this is mostly well-known but is included here for lack of an adequate reference. Appendix B is a discussion of the linear algebra of modules over Gorenstein rings. We also give an introduction to the theory of bilateral derivations.

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# Notation and terminology

## Fields

Any time a field $F$ appears in the text we assume that it is accompanied by a fixed choice of separable algebraic closure $F_{\mathrm{s}}$. We further assume that for any inclusion of fields $F \hookrightarrow F'$ these choices are made in such a way that $F_{\mathrm{s}}$ is a subfield of $F'_{\mathrm{s}}$. We write $G_F$ for the absolute Galois group $\mathrm{Gal}(F_{\mathrm{s}}/F)$ of $F$ and $\operatorname{cd} F$ for the Galois cohomological dimension of $F$. By a *global field* we will mean a finite extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$ (for some prime $p$) and by a *local field* we will mean a finite extension of $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$ (for some prime $p$).

If $k$ is a finite field, by a *Frobenius element* $\mathrm{Fr}$ for $k$ we will always mean a geometric Frobenius element, normalized by $\mathrm{Fr}(x)^{\#k} = x$ for all $x \in k$. If $X$ is a scheme over $k$, we let $\mathrm{Fr}$ denote the $k$-linear Frobenius morphism of $X$. If $F$ is a global field and $v$ is a place of $F$, we write $\mathrm{Fr}(v)$ for a geometric Frobenius element of $G_F$.

## Characters

For any field $F$ of characteristic different from $l$, we let $\mathbf{Z}_l(1)$ denote the Tate module of the $l$-power roots of unity. The natural map $G_F \to \operatorname{Aut}_{\mathbf{Z}_l} \mathbf{Z}_l(1)$ will be called the *cyclotomic character* and denoted $\varepsilon$. We write $\mathbf{Z}_l(-1)$ for the dual $\operatorname{Hom}_{\mathbf{Z}_l}(\mathbf{Z}_l(1), \mathbf{Z}_l)$ of $\mathbf{Z}_l(1)$.

Let $F$ be a global field and $\chi : G_F \to A^\times$ a character. If $v$ is a place of $F$ at which $\chi$ is unramified, we write $\chi(v)$ for $\chi(\mathrm{Fr}(v))$. Note that $\varepsilon(\mathrm{Fr}(v)) = (\#k_v)^{-1}$.

## Galois modules

If $F$ is a field and $T$ is a topological abelian group with an action of $G_F$, we will always assume that the $G_F$-action is continuous for the profinite topology on $G_F$ and the given topology on $T$. If $T$ is such a $G_F$-module, we write $F(T)$ for the fixed field of the kernel of the map $G_F \to \operatorname{Aut} T$; we call $F(T)$ the *splitting field* of $T$ over $F$. If $T$ is a $\mathbf{Z}_l$-module with a $\mathbf{Z}_l$-linear action of $G_F$ and $n > 0$, then we write $T(n)$ for the $n$-fold tensor product of $T$ with $\mathbf{Z}_l(1)$; if $n < 0$, we write $T(n)$ for the $|n|$-fold tensor product of $T$ with $\mathbf{Z}_l(-1)$.

## Schemes

If $x$ is a point of a scheme $X$, we write $k(x)$ for its residue field and define its *codimension*, denoted $\operatorname{codim}_X x$, to be the dimension of the local ring $\mathcal{O}_{X,x}$. If $Z$ is an arbitrary closed subset of $X$, we define $k(Z)$ to be the product of the residue fields of the minimal points of $Z$, and we define $\operatorname{codim}_X Z$ to be the least codimension of any point of $Z$. We write $X^p$ for the set of points of $X$ of codimension exactly $p$. If $x$ is a point of $X$, we will write $\bar{x}$ for the reduced closed subscheme of $X$ defined on the closure of $\{x\}$. If $X$ is a scheme over a base $S$ and $S' \to S$ is any morphism, we will write $X_{S'}$ for the fibre product $X \times_S S'$. If $Y$ and $Z$ are two subschemes of a third scheme $X$, then by the *scheme-theoretic intersection* of $Y$ and $Z$ (in $X$) we mean the fibre product $Y \times_X Z$.

By a *variety* over a field $F$ we will mean a reduced irreducible separated scheme of finite type over $\operatorname{Spec} F$. If $X$ and $Y$ are varieties over a field $F$, then we write $X \times Y$ for the fiber product $X \times_{\operatorname{Spec} F} Y$.

### Sheaves

All sheaves other than structure sheaves are assumed to be sheaves for the étale topology unless otherwise specified; locally constant sheaves are assumed to be locally constant for the étale topology. If $i : U \to X$ is an open immersion and $\mathcal{F}$ is a sheaf on $X$, we will usually just write $\mathcal{F}$ for the pullback sheaf $i^*\mathcal{F}$ on $U$. If $\mathcal{F}$ is any torsion sheaf, we write $\mathcal{F}(m)$ for its $m^{\text{th}}$ Tate twist:

$$\mathcal{F}(m) = \mathcal{F} \otimes \mu_\infty^{\otimes m}.$$

If $i : X' \to X$ is a morphism and $\mathcal{F}$ is a Tate twist of a constant sheaf on $X$ we will usually still write $\mathcal{F}$ for the pullback sheaf $i^*\mathcal{F}$ on $X'$.

### Cohomology

All spectral sequences are assumed to be cohomological. All cohomology is either étale or Galois; we will attempt to be careful as to which is which, even when they coincide. If $L/K$ is an extension of fields and $T$ is a topological $\operatorname{Gal}(L/K)$-module, we write $H^i(L/K, T)$ for the cohomology group $H^i(\operatorname{Gal}(L/K), T)$, computed with continuous cochains. If $L$ is a separable algebraic closure of $K$, then we just write $H^i(K, T)$ for these cohomology groups.

If $\mathcal{F}$ is an $l$-adic étale sheaf on a scheme $X$, then we write $H^i(X, \mathcal{F})$ for the inverse limit of the étale cohomology groups $H^i(X, \mathcal{F}/l^n\mathcal{F})$. If $\mathcal{F}$ is an étale sheaf of $\mathbf{Q}_l$-vector spaces, then we write $H^i(X, \mathcal{F})$ for the tensor product of $\mathbf{Q}_l$ with $H^i(X, \mathcal{F}_0/l^n\mathcal{F}_0)$ where $\mathcal{F}_0$ is any $l$-adic subsheaf of $\mathcal{F}$ with $\mathcal{F}_0 \otimes \mathbf{Q}_l = \mathcal{F}$; this definition is independent of the choice of $\mathcal{F}_0$.

### $K$-theory

We write $K_i$ and $K_i'$ for Quillen's $K$-groups for the category of locally free sheaves and the category of all coherent sheaves respectively; these functors agree on regular schemes. We will write $\mathcal{K}_i(X)$ for the Zariski sheaf of $K$-groups on $X$. We take all $K$-groups to vanish for negative indices.

# Part 1

# Selmer groups and deformation theory

CHAPTER 1

# Local cohomology groups

In this chapter we give the basic theory of finite/singular structures over local fields in preparation for the definition of Selmer groups in Chapter II.

## 1. Local finite/singular structures

Fix a prime $l$ and let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra. Let $K$ be a local field with residue field $k$ of characteristic $p$; we allow $p = l$. We write $K_{\mathrm{ur}}$ for the maximal unramified extension of $K$, and we let $\mathcal{I}_K = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{ur}})$ denote the inertia group of $K$.

Let $T$ be an $A$-module with an $A$-linear action of $G_K$. We further assume that one of the following holds:

- $T$ is a finitely generated $\mathbf{Z}_l$-module and the $G_K$-action on $T$ is continuous for the $l$-adic topology on $T$; or
- $T$ is a torsion $\mathbf{Z}_l$-module of finite corank (that is, $T$ is isomorphic as a $\mathbf{Z}_l$-module to $(\mathbf{Q}_l/\mathbf{Z}_l)^r \oplus T'$ for some $r \geq 0$ and some $\mathbf{Z}_l$-module $T'$ of finite order) and the $G_K$-action on $T$ is continuous for the discrete topology on $T$.

We will be working with cohomology with continuous cochains (see [**Rub00**, Appendix B]) and these assumptions are necessary in order to insure that it is well behaved. In the second case, continuous cohomology agrees with the usual profinite/discrete cohomology. We will refer to $T$ as above as *$l$-adic $G_K$-modules over $A$*; if $T$ satisfies the first condition we will say that it is *finitely generated*, and if it satisfies the second condition we will say that it is *discrete*. Note that $T$ is both finitely generated and discrete if and only if $T$ is finite.

We require maps of $l$-adic $G_K$-modules over $A$ to be continuous, $A$-linear and $G_K$-equivariant. (In fact, the continuity is a consequence of the $A$-linearity.) We will say that $T$ is *unramified* if $\mathcal{I}_K$ acts trivially on $T$.

Definition 1.1. The *unramified subgroup* $H^1_{\mathrm{ur}}(K,T)$ of the $K$-cohomology of $T$ is

$$H^1_{\mathrm{ur}}(K,T) = \ker\bigl(H^1(K,T) \to H^1(K_{\mathrm{ur}},T)\bigr).$$

[**Rub00**, Lemma 1.3.2] identifies $H^1_{\mathrm{ur}}(K,T)$ with $H^1(K_{\mathrm{ur}}/K, T^{\mathcal{I}_K})$ via inflation, and this further identifies with $H^1(k, T^{\mathcal{I}_K})$. Note also that $H^1_{\mathrm{ur}}(K,T)$ is naturally an $A$-module since the action of $G_K$ on $T$ is $A$-linear.

Definition 1.2. A *local finite/singular structure* $\mathcal{S}$ on $T$ consists of a choice of $A$-submodule $H^1_{f,\mathcal{S}}(K,T) \subseteq H^1(K,T)$.

We will write $H^1_{s,\mathcal{S}}(K,T)$ for the $A$-module quotient $H^1(K,T)/H^1_{f,\mathcal{S}}(K,T)$. We write $c_s$ for the image of a cohomology class $c \in H^1(K,T)$ under the quotient map

$H^1(K,T) \to H^1_{s,\mathcal{S}}(K,T)$. We call $H^1_{f,\mathcal{S}}(K,T)$ and $H^1_{s,\mathcal{S}}(K,T)$ the *finite subgroup* and the *singular subgroup* of the $K$-cohomology of $T$ respectively. We will omit the structure $\mathcal{S}$ from the notation if it is clear from context.

The standard choice for $H^1_{f,\mathcal{S}}(K,T)$ (at least when $p \neq l$) is the unramified subgroup $H^1_{\mathrm{ur}}(K,T)$. In this case we have the following description of $H^1_{s,\mathcal{S}}(K,T)$.

LEMMA 1.3. *Let $T$ be an $l$-adic $G_K$-module over $A$. Assume that $p \neq l$ and that $T$ is unramified. Let $\mathcal{S}$ be the local finite/singular structure on $T$ given by $H^1_{f,\mathcal{S}}(K,T) = H^1_{\mathrm{ur}}(K,T)$. Then $H^1_{s,\mathcal{S}}(K,T) \cong T(-1)^{G_k}$.*

PROOF. We can write the inflation-restriction exact sequence as

$$0 \to H^1(k,T) \to H^1(K,T) \to H^1(\mathcal{I}_K,T)^{G_k} \to H^2(k,T).$$

Since $k$ has cohomological dimension 1 [**Ser97**, Chapter 2, Section 3], the last term vanishes. It follows that

$$H^1_{s,\mathcal{S}}(K,T) \cong H^1(\mathcal{I}_K,T)^{G_k}.$$

Since $T$ is unramified there is also an isomorphism

$$H^1(\mathcal{I}_K,T)^{G_k} \cong \mathrm{Hom}_{G_k}(\mathcal{I}_K,T).$$

$T$ is a pro-$l$ group, so any homomorphism $\mathcal{I}_K \to T$ must factor through the maximal pro-$l$ quotient of $\mathcal{I}_K$. Letting $\pi$ denote a uniformizer of $K$, this quotient is $\mathrm{Gal}(K_{\mathrm{ur}}(\pi^{1/l^\infty})/K_{\mathrm{ur}})$, which as a $G_k$-module identifies with $\mathbf{Z}_l(1)$; see [**Frö67**, Section 8]. Thus

$$\mathrm{Hom}_{G_k}(\mathcal{I}_K,T) \cong \mathrm{Hom}_{G_k}(\mathbf{Z}_l(1),T) \cong T(-1)^{G_k}$$

as claimed. □

## 2. Functorialities

Let $f: T \to T'$ be a map of $l$-adic $G_F$-modules over $A$. Assume also that $T$ and $T'$ have local finite/singular structures $\mathcal{S}$ and $\mathcal{S}'$ respectively. Let

$$f_*: H^1(K,T) \to H^1(K,T')$$

denote the map induced by $f$. We say that the structures $\mathcal{S}$, $\mathcal{S}'$ are *compatible with* $f$ if

$$f_* H^1_{f,\mathcal{S}}(K,T) \subseteq H^1_{f,\mathcal{S}'}(K,T').$$

If this is the case, then there are natural maps

$$H^1_{f,\mathcal{S}}(K,T) \to H^1_{f,\mathcal{S}'}(K,T')$$
$$H^1_{s,\mathcal{S}}(K,T) \to H^1_{s,\mathcal{S}'}(K,T').$$

Note that unramified structures are always compatible.

Let $i: T' \hookrightarrow T$ and $j: T \twoheadrightarrow T''$ be an injection and a surjection of $l$-adic $G_K$-modules over $A$, respectively. Given a local finite/singular structure $\mathcal{S}$ on $T$, we define the *induced local finite/singular structures* $i^*\mathcal{S}$ and $j_*\mathcal{S}$ on $T'$ and $T''$ by

$$H^1_{f,i^*\mathcal{S}}(K,T') = i_*^{-1} H^1_{f,\mathcal{S}}(K,T)$$
$$H^1_{f,j_*\mathcal{S}}(K,T'') = j_* H^1_{f,\mathcal{S}}(K,T).$$

One checks easily that these structures are compatible with $i$ and $j$, respectively. We will usually just write $\mathcal{S}$ for $i^*\mathcal{S}$ or $j_*\mathcal{S}$ if the maps are clear from context.

LEMMA 2.1. *Let* $i : T' \hookrightarrow T$ *and* $j : T \twoheadrightarrow T''$ *be maps of unramified* $l$*-adic* $G_K$*-modules over* $A$. *Let* $\mathcal{S}$ *denote the unramified finite/singular structure on* $T$. *Then* $i^*\mathcal{S}$ *(resp.* $j_*\mathcal{S}$*) is the unramified structure on* $T'$ *(resp.* $T''$*).*

PROOF. This is an easy diagram chase; the proof for $j_*\mathcal{S}$ requires the fact that $k$ has cohomological dimension 1. □

## 3. Local exact sequences

Let $T$ be an $l$-adic $G_K$-module over $A$ with a given local finite/singular structure $\mathcal{S}$. Let

$$0 \to T' \to T \to T'' \to 0$$

be an exact sequence of $G_K$-modules, and give $T'$ and $T''$ the local finite/singular structures induced from $\mathcal{S}$. In this situation the long exact sequence of $G_K$-cohomology splits into a "finite" and a "singular" exact sequence.

LEMMA 3.1. *Let* $0 \to T' \to T \to T'' \to 0$ *be an exact sequence of* $l$*-adic* $G_K$*-modules over* $A$. *Let* $T$ *have a finite/singular structure* $\mathcal{S}$ *and let* $T'$ *and* $T''$ *have the induced structures. Then there are exact sequences*

$$0 \longrightarrow H^0(K,T') \longrightarrow H^0(K,T) \longrightarrow H^0(K,T'') \longrightarrow$$
$$H^1_f(K,T') \longrightarrow H^1_f(K,T) \longrightarrow H^1_f(K,T'') \longrightarrow 0$$

*and*

$$0 \longrightarrow H^1_s(K,T') \longrightarrow H^1_s(K,T) \longrightarrow H^1_s(K,T'') \longrightarrow$$
$$H^2(K,T') \longrightarrow H^2(K,T) \longrightarrow H^2(K,T'') \longrightarrow 0$$

PROOF. We begin with the long exact sequence of $G_K$-cohomology

$$0 \longrightarrow H^0(K,T') \longrightarrow H^0(K,T) \longrightarrow H^0(K,T'') \longrightarrow \tag{3.1}$$
$$H^1(K,T') \longrightarrow H^1(K,T) \longrightarrow H^1(K,T'')$$

Since $T'$ and $T''$ have the induced finite/singular structures we have a canonical sequence

$$H^1_f(K,T') \to H^1_f(K,T) \to H^1_f(K,T''). \tag{3.2}$$

The map $H^0(K,T'') \to H^1_f(K,T')$ comes from the exactness of

$$H^0(K,T'') \to H^1(K,T') \to H^1(K,T)$$

and the fact that $H^1_f(K,T')$ contains the full inverse image of $0 \in H^1(K,T)$. Combining this with (3.1) and (3.2) yields the first sequence of the lemma; exactness is easily checked using the exactness of (3.1) and the definition of induced structures.

For the second exact sequence, we begin with the exact sequence

$$H^1(K,T') \longrightarrow H^1(K,T) \longrightarrow H^1(K,T'') \longrightarrow$$
$$H^2(K,T') \longrightarrow H^2(K,T) \longrightarrow H^2(K,T'') \longrightarrow 0$$

The last map is a surjection by standard cohomological dimension results; see [**Ser97**, Section 5.3, Proposition 15]. The existence of the sequence

$$H^1_s(K,T') \to H^1_s(K,T) \to H^1_s(K,T'')$$

follows immediately from the compatibility of the finite/singular structures, and the map $H^1_s(K,T'') \to H^2(K,T')$ exists since $H^1_f(K,T'')$ is the image of $H^1_f(K,T)$ and thus is in the kernel of $H^1(K,T'') \to H^2(K,T')$. This yields the sequence, and exactness is checked by an easy diagram chase and the fact that the map $H^1_f(K,T) \to H^1_f(K,T'')$ is surjective. □

## 4. Examples of local structures

Following Bloch, Kato and others, we will consider several different choices of local finite/singular structures, depending on the behavior of $T$ as an $\mathcal{I}_K$-module.

**$T$ arbitrary, $p$ arbitrary:** The *weak structure* is given by

$$H^1_f(K,T) = H^1(K,T).$$

**$T$ arbitrary, $p$ arbitrary:** The *strong structure* is given by

$$H^1_f(K,T) = 0.$$

**$T$ unramified, $p \neq l$:** The *unramified structure* is given by

$$H^1_f(K,T) = H^1_{\mathrm{ur}}(K,T).$$

For the rest of the definitions we first must define a certain $\mathbf{Q}_l$-vector space $V$. Assume for this that $T$ is free over $\mathbf{Z}_l$ (resp. is $l$-divisible). If $T$ is finitely-generated (resp. discrete), then set $V = T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ (resp. $V = (\varprojlim T[l^n]) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$). We will define choices of $H^1_f(K,V)$; these give rise to corresponding choices of $H^1_f(K,T)$ by pulling back via the natural map $T \hookrightarrow V$ (resp. pushing forward via the natural map $V \twoheadrightarrow T$).

**$T$ as above, $p \neq l$:** The *minimally ramified structure* is given by

$$H^1_f(K,V) = H^1_{\mathrm{ur}}(K,V).$$

(One checks as in Lemma 2.1 that this yields the unramified structure if $T$ is unramified.)

**$T$ as above, $p = l$:** The *exponential structure* is given by

$$H^1_f(K,V) = \ker\big(H^1(K,V) \to H^1(K, V \otimes_{\mathbf{Q}_l} B_{\mathrm{cris}}^{f=1})\big),$$

where $f$ is the Frobenius endomorphism of $B_{\mathrm{cris}}$.

**$T$ as above, $p = l$:** The *crystalline structure* is given by

$$H^1_f(K,V) = \ker\big(H^1(K,V) \to H^1(K, V \otimes_{\mathbf{Q}_l} B_{\mathrm{cris}})\big).$$

**$T$ as above, $p = l$:** The *deRham structure* is given by

$$H^1_f(K,V) = \ker\big(H^1(K,V) \to H^1(K, V \otimes_{\mathbf{Q}_l} B_{\mathrm{dR}})\big).$$

Here $B_{\mathrm{cris}}$ and $B_{\mathrm{dR}}$ are the "big rings" of Fontaine; see [**FI93**] for an exposition and references.

Of course, if $T$ arises as a quotient of a free $\mathbf{Z}_l$-module or as a subgroup of an $l$-divisible $\mathbf{Z}_l$-module, then one can give $T$ a local finite/singular structure induced by one of the above structures.

## 5. Ordinary representations

It will be useful to give an "explicit" characterization of the minimally ramified finite/singular structure on certain ramified rank 3 representations. Assume for this section that $K$ does not have characteristic $l$.

DEFINITION 5.1. Let $T$ be an $l$-adic $G_K$-module over $A$ which is free of rank 2 as an $A$-module. We say that $T$ is *ordinary* if $\mathcal{I}_K$ acts non-trivially on $T$ and if there is an exact sequence

$$0 \to A(1) \to T \to A \to 0$$

which is $G_L$-equivariant for some unramified extension $L$ of $K$.

LEMMA 5.2. *Let $T$ be an ordinary representation. Then the minimally ramified and weak structures on* $\operatorname{End}^0_A T(1)$ *coincide.*

PROOF. Set $B = A \otimes \mathbf{Q}_l$ and $V = T \otimes \mathbf{Q}_l$; by the definition of the minimally ramified structure, we must show that

$$H^1_{\mathrm{ur}}(K, \operatorname{End}^0_B V(1)) = H^1(K, \operatorname{End}^0_B V(1)).$$

By the inflation-restriction exact sequence, to show this it suffices to show that

$$H^1(\mathcal{I}_K, \operatorname{End}^0_B V(1))^{G_k} = 0. \tag{5.1}$$

Let us recall some facts about the cohomology of $\mathcal{I}_K$. First, it has cohomological dimension 1; see [**Ser97**, Chapter 2, Section 3.3(c)]. Secondly, by [**Frö67**, Section 8] the maximal pro-$l$ quotient of $\mathcal{I}_K$ is isomorphic to $\mathbf{Z}_l(1)$ as a $G_k$-module; since $B(i)$ is unramified for any $i$, it follows that there is an isomorphism

$$H^1(\mathcal{I}_K, B(i)) \cong \operatorname{Hom}(\mathcal{I}_K, B(i)) \cong \operatorname{Hom}_{\mathbf{Z}_l}(\mathbf{Z}_l(1), B(i)) \cong B(i-1)$$

of $G_k$-modules.

Since $T$ is ordinary, there is a $B$-linear filtration

$$0 \to B(1) \to V \to B \to 0 \tag{5.2}$$

which is $G_L$-equivariant, where $L$ is a finite unramified extension of $K$. In particular, $\mathcal{I}_K$ is also the inertia group of $L$.

By (5.2) we can choose a basis $x, y$ of $V$ such that

$$\gamma x = \varepsilon(\gamma) x$$
$$\gamma y = y + \nu(\gamma) y$$

for all $\gamma \in G_L$; here $\varepsilon$ is the cyclotomic character and $\nu : G_L \to B$ is some map. By definition $V$ is actually ramified, so we know that $\nu(\mathcal{I}_K) \neq 0$.

Twisting (5.2) by $B(1)$ and taking $\mathcal{I}_K$-cohomology yields an exact sequence

$$0 \to B(2) \xrightarrow{\alpha_1} V(1)^{\mathcal{I}_K} \xrightarrow{\alpha_2} B(1) \xrightarrow{\alpha_3} B(1) \xrightarrow{\alpha_4} H^1(\mathcal{I}_K, V(1)) \xrightarrow{\alpha_5} B \to 0.$$

Using our basis of $V$, one finds that $V(1)^{\mathcal{I}_K} \cong B(2)$, so $\alpha_1$ is an isomorphism. Thus $\alpha_2$ is the zero map, so $\alpha_3$ is also an isomorphism. Now $\alpha_4 = 0$, so $\alpha_5$ is an isomorphism. We conclude that $H^1(\mathcal{I}_K, V(1)) \cong B$.

Using our basis of $V$ one can compute $\operatorname{End}^0_B V(1)$ completely explicitly; one finds a $G_L$-equivariant filtration

$$0 \to V(1) \to \operatorname{End}^0_B V(1) \to B \to 0.$$

The long exact sequence in $\mathcal{I}_K$-cohomology and our computations above yield an exact sequence

$$0 \to B(2) \xrightarrow{\beta_1} (\mathrm{End}_B^0 V(1))^{\mathcal{I}_K} \xrightarrow{\beta_2} B \xrightarrow{\beta_3} B \xrightarrow{\beta_4} H^1(\mathcal{I}_K, \mathrm{End}_B^0 V(1)) \xrightarrow{\beta_5} B(-1) \to 0.$$

One computes directly that $(\mathrm{End}_B^0 V(1))^{\mathcal{I}_K} = B(2)$, so $\beta_1$ is an isomorphism. Thus $\beta_2 = 0$ and $\beta_3$ is an isomorphism. Now $\beta_4 = 0$, so

$$H^1(\mathcal{I}_K, \mathrm{End}_B^0 V(1)) \cong B(-1)$$

as $G_L$-modules. $B(-1)$ has no $G_L$-invariants, so this yields (5.1) as desired. □

## 6. Cartier dual structures

Let $T$ be an $l$-adic $G_K$-module over $A$. We define the *Cartier dual* $T^*$ of $T$ to be $\mathrm{Hom}_{\mathbf{Z}_l}(T, \mu_{l^\infty}(K_{\mathrm{s}}))$ with the induced $A$-module structure (via the $A$-module structure on $T$). We give $T^*$ a $G_K$-action by ${}^g f(t) = gf(g^{-1}t)$ for $f \in T^*$, $g \in G_K$ and $t \in T$. $T^*$ is also an $l$-adic $G_K$-module over $A$; if $T$ is finitely generated, then $T^*$ will be discrete, and if $T$ is discrete, then $T^*$ will be finitely generated. For any ideal $\mathfrak{a}$ of $A$, there are canonical identifications:

$$\begin{aligned}(T/\mathfrak{a}T)^* &\cong T^*[\mathfrak{a}]\\ T[\mathfrak{a}]^* &\cong T^*/\mathfrak{a}T^*\\ (\mathfrak{a}T)^* &\cong T^*/T^*[\mathfrak{a}]\\ (T/T[\mathfrak{a}])^* &\cong \mathfrak{a}T^*.\end{aligned}$$

One easily checks that $(T^*)^*$ is canonically isomorphic to $T$.

The cohomology groups of $T$ and $T^*$ are related by Tate local duality.

THEOREM 6.1 (Tate local duality). *Cup product, Cartier duality and the invariant map of local class field theory yield a perfect $A$-hermitian pairing*

$$H^i(K, T) \otimes_{\mathbf{Z}_l} H^{2-i}(K, T^*) \to H^2(K, T \otimes_{\mathbf{Z}_l} T^*) \to H^2(K, \mu_{l^\infty}) \xrightarrow{\simeq} \mathbf{Q}_l/\mathbf{Z}_l.$$

*Furthermore, if $p \neq l$ and $T$ is unramified, then $H^1_{\mathrm{ur}}(K, T)$ and $H^1_{\mathrm{ur}}(K, T^*)$ are exact orthogonal complements under this pairing with $i = 1$.*

PROOF. See [**Mil86**, Corollary I.2.3] and [**Rub00**, Chapter 1, Section 4]. Rubin proves his result only for the characteristic 0 case and does not state it in exactly this form, but the general case is the same; one replaces his dimension counts with rank counts in the free case and cardinality counts in the finite case and then combines the two results. The fact that the pairing is $A$-hermitian is clear since $G_K$ acts $A$-linearly. □

Given a local finite/singular structure $\mathcal{S}$ on $T$, we can use Theorem 6.1 to define a local finite/singular structure $\mathcal{S}^*$ on $T^*$: we define $H^1_{f,\mathcal{S}^*}(K, T^*)$ to be the exact orthogonal complement of $H^1_{f,\mathcal{S}}(K, T)$ under Tate local duality. We call this the *Cartier dual local finite/singular structure* on $T^*$. Tate local duality restricts to yield perfect pairings

$$\begin{aligned}H^1_{f,\mathcal{S}}(K, T) \otimes_{\mathbf{Z}_l} H^1_{s,\mathcal{S}^*}(K, T^*) &\to \mathbf{Q}_l/\mathbf{Z}_l\\ H^1_{s,\mathcal{S}}(K, T) \otimes_{\mathbf{Z}_l} H^1_{f,\mathcal{S}^*}(K, T^*) &\to \mathbf{Q}_l/\mathbf{Z}_l.\end{aligned}$$

If $\mathcal{S}$ is the weak (resp. strong, resp. minimally ramified) structure, then $\mathcal{S}^*$ is the strong (resp. weak, resp. minimally ramified) structure; see [**Rub00**, Chapter

1, Section 4]. In particular, if $T$ is unramified and $\mathcal{S}$ is the unramified structure, then $\mathcal{S}^*$ is also the unramified structure. To make similar statements for the more subtle structures when $p = l$, we need to assume that $T$ is free over $\mathbf{Z}_l$ and that $T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is deRham. In this case, if $\mathcal{S}$ is the crystalline (resp. exponential, resp. deRham) structure, then $\mathcal{S}^*$ is the crystalline (resp. deRham, resp. exponential) structure; see [**BK90**, Proposition 3.8].

## 7. Local structures for archimedean fields

We now consider the archimedean case. Let $K$ denote either $\mathbf{R}$ or $\mathbf{C}$ and let $T$ be a $G_K$-module. If $K = \mathbf{C}$, then $H^1(K, T)$ is trivial, so there is no structure to assign. If $K = \mathbf{R}$ and $l \neq 2$, the same is true of $H^1(K, T)$, as it is 2-torsion and 2 is invertible acting on $T$. The only interesting case is $K = \mathbf{R}$ and $l = 2$. In this case one can define weak and strong structures as before. If $T$ is free or divisible, we can also attempt to define a minimally ramified structure. However, since $H^1(K, V) = 0$, there is no choice for the structure on $V$. Thus this just gives rise to the weak (resp. strong) structure on $T$ which are finitely generated (resp. discrete).

CHAPTER 2

# Global cohomology groups

We begin this chapter by defining global finite/singular structures and Selmer groups. We then turn to the definitions of the local/global Kolyvagin pairing and the global Bockstein pairing. The Kolyvagin pairing will be of fundamental importance to the annihilation theorems of Chapter III, while the Bockstein pairing is of independent interest and will be studied more closely in Chapter V.

## 1. Selmer groups

Let $F$ be a global field and let $M_F$ denote the set of places of $F$. Recall that if $F$ is a number field, then $M_F$ consists of both archimedean and non-archimedean places, while if $F$ is a function field, then $M_F$ consists only of non-archimedean places; see [**Cas67**, Section 3 and Section 12] for details. For every place $v$ we fix now and forever embeddings $F_{\mathrm{s}} \hookrightarrow F_{v,\mathrm{s}}$; these induce injections $G_{F_v} \hookrightarrow G_F$, and changing the choice of embedding changes these injections by conjugation. Let $k_v$ denote the residue field of $F_v$ and let $\mathcal{I}_v = \mathrm{Gal}(F_{v,\mathrm{s}}/F_{v,\mathrm{ur}})$ denote the inertia group of $F_v$.

Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra with maximal ideal $\mathfrak{m}$ and residue field $k$. We assume that $l$ does not equal the characteristic of $F$. Let $\Sigma_l$ denote the set of places of $F$ above $l$; $\Sigma_l$ is empty if $F$ is a function field.

Definition 1.1. An *$l$-adic $G_F$-module over $A$* is an $A$-module $T$ endowed with an $A$-linear action of $G_F$ such that the action of $G_{F_v}$ on $T$ is unramified for almost all $v$ and such that one of the following holds:

- $T$ is a finitely generated $\mathbf{Z}_l$-module and the $G_F$-action on $T$ is continuous for the $l$-adic topology on $T$; or
- $T$ is a torsion $\mathbf{Z}_l$-module of finite corank (that is, $T$ is isomorphic as a $\mathbf{Z}_l$-module to $(\mathbf{Q}_l/\mathbf{Z}_l)^r \oplus T'$ for some $r \geq 0$ and some $\mathbf{Z}_l$-module $T'$ of finite order) and the $G_F$-action on $T$ is continuous for the discrete topology on $T$.

In the first case we say that $T$ is *finitely generated* and in the second case we say that $T$ is *discrete*.

We will say that a set of places $\Sigma$ of $F$ is *sufficiently large for $T$* if it contains $\Sigma_l$, all archimedean places and all places where $T$ is ramified; by the definition of $l$-adic $G_F$-modules, there exist finite sets of places which are sufficiently large for $T$.

For every place $v$ there is a canonical restriction map

$$\mathrm{res}_v : H^1(F,T) \to H^1(F_v,T);$$

$\mathrm{res}_v$ is initially determined by our embedding $F_v \hookrightarrow F_{v,\mathrm{s}}$, but by [**Ser79**, Chapter 7, Proposition 3] is actually independent of this choice. If $c \in H^1(F,T)$, then we write

$c_v$ for its image under $\mathrm{res}_v$. We have the following fundamental lemma regarding these maps.

LEMMA 1.2. *Let $T$ be a discrete $l$-adic $G_F$-module over $A$ and let $c \in H^1(F,T)$ be a cohomology class. Then $c_v$ lies in $H^1_{\mathrm{ur}}(F_v,T)$ for almost all $v$.*

PROOF. Let $\tilde{c} : G_F \to T$ be a cocycle representing $c$; since $T$ is discrete as a $G_F$ module and $G_F$ is compact, there is some finite extension $F'$ of $F$ such that $\tilde{c}$ factors through $\mathrm{Gal}(F'/F)$. Now let $\Sigma$ be a finite set of places of $F$ containing all archimedean places and all places where $F'/F$ is ramified. $\tilde{c}_v : G_{F_v} \to T$ factors through $\mathrm{Gal}(F'_v/F_v)$; this extension is unramified away from $\Sigma$, so so $c_v$ is an unramified cocycle for $v \notin \Sigma$. This proves the lemma. □

The global analogue of a local finite/singular structure is given by specifying local finite/singular structures at every place.

DEFINITION 1.3. Let $T$ be an $l$-adic $G_F$-module over $A$. A *finite/singular structure* $\mathcal{S}$ on $T$ consists of choices of local finite/singular structures $H^1_{f,\mathcal{S}}(F_v,T)$ for all places $v$ of $F$ such that $H^1_{f,\mathcal{S}}(F_v,T) = H^1_{\mathrm{ur}}(F_v,T)$ for almost all $v$.

Let $\Sigma$ be a finite set of places of $F$. We will say that a finite/singular structure $\mathcal{S}$ is *unramified away from a set of places* $\Sigma$ if the local finite/singular structures at $v$ are unramified for $v \notin \Sigma$.

If $T$ is free over $\mathbf{Z}_l$ or $l$-divisible, then the structures considered in [**BK90**] and [**FPR94**] are those which are minimally ramified away from $\Sigma_l$; they consider various possibilities for the structures at $\Sigma_l$.

A finite/singular structure determines a Selmer group, which will be our central object of study.

DEFINITION 1.4. The *Selmer group* $H^1_{f,\mathcal{S}}(F,T)$ of $T$ (with the finite/singular structure $\mathcal{S}$) is the kernel of the map

$$H^1(F,T) \to \prod_{v \in M_F} H^1_{s,\mathcal{S}}(F_v,T);$$

that is,

$$H^1_{f,\mathcal{S}}(F,T) = \{c \in H^1(F,T) \mid c_v \in H^1_{f,\mathcal{S}}(F_v,T) \text{ for all } v\},$$

the set of global cohomology classes which are everywhere locally finite.

See [**Rub00**, Chapter 1, Section 6] for interpretations of Selmer groups in terms of ideal class groups, global units and rational points on abelian varieties.

DEFINITION 1.5. The *Kolyvagin group* $H^1_{s,\mathcal{S}}(F,T)$ is defined to be the quotient $H^1(F,T)/H^1_{f,\mathcal{S}}(F,T)$.

There is a natural map

$$H^1_{s,\mathcal{S}}(F,T) \to \prod_{v \in M_F} H^1_{s,\mathcal{S}}(F_v,T);$$

if $T$ is a discrete $G_F$-module, then Lemma I.1.2 shows that the image of this map actually lands in $\oplus_v H^1_{s,\mathcal{S}}(F_v,T)$.

## 2. Functorialities

Let $f : T \to T'$ be a map of $l$-adic $G_F$-modules over $A$. Assume also that $T$ and $T'$ have finite/singular structures $\mathcal{S}$ and $\mathcal{S}'$ respectively. We say that these structures are *compatible with* $f$ if the local finite/singular structures at $F_v$ are compatible with $f$ for every place $v$ of $F$; in this case there is an induced map

$$H^1_{f,\mathcal{S}}(F,T) \to H^1_{f,\mathcal{S}'}(F,T')$$

of Selmer groups.

Let $i : T' \hookrightarrow T$ and $j : T \twoheadrightarrow T''$ be an injection and a surjection of $l$-adic $G_F$-modules over $A$, respectively. Given a finite/singular structure $\mathcal{S}$ on $T$, we define the *induced finite/singular structures* $i^*\mathcal{S}$ and $j_*\mathcal{S}$ on $T'$ and $T''$ by assigning the induced local finite/singular structures for every place $v$. By Lemma I.2.1 $i^*\mathcal{S}$ and $j_*\mathcal{S}$ really are unramified almost everywhere, as required, and they are visibly compatible with $\mathcal{S}$. This construction applies in particular to maps of the form $T[\mathfrak{a}] \hookrightarrow T$ and $T \twoheadrightarrow T/\mathfrak{a}T$, where $\mathfrak{a}$ is an ideal of $A$; we will always assume that finite/singular structures on such modules are induced as above. We will usually just write $\mathcal{S}$ for $i^*\mathcal{S}$ or $j_*\mathcal{S}$ if the maps are clear from context.

Definition 2.1. If $T$ is an $l$-adic $G_F$-module over $A$ with a finite/singular structure $\mathcal{S}$, we define the *Cartier dual* $T^*$ of $T$ to be the $l$-adic $G_F$-module over $A$ $\mathrm{Hom}_{\mathbf{Z}_l}(T, \mu_{l^\infty})$ with a finite/singular structure $\mathcal{S}^*$ given by the local Cartier dual finite/singular structure at every place of $F$.

Note that Theorem I.6.1 and our assumption that $T$ is ramified at only finitely many places insures that the structure $\mathcal{S}^*$ really is unramified almost everywhere.

## 3. The global exact sequence

In this section we give the global analogue of the first local exact sequence of Lemma I.3.1.

Lemma 3.1. *Let* $0 \to T' \to T \to T'' \to 0$ *be an exact sequence of* $l$*-adic* $G_F$*-modules over* $A$*. Let* $\mathcal{S}$ *be a finite/singular structure on* $T$ *and let* $T'$ *and* $T''$ *have the induced finite/singular structures. Then there is an exact sequence*

$$0 \to H^0(F,T') \to H^0(F,T) \to H^0(F,T'') \to H^1_f(F,T') \to H^1_f(F,T) \to H^1_f(F,T'')$$

Proof. Exactness at the $H^0$-terms follows from the long exact sequence in $G_F$-cohomology. The existence and exactness of the remaining maps follows from the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(F,T'') & \dashrightarrow & H^1_f(F,T') & \dashrightarrow & H^1_f(F,T) & \dashrightarrow & H^1_f(F,T'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(F,T'') & \longrightarrow & H^1(F,T') & \longrightarrow & H^1(F,T) & \longrightarrow & H^1(F,T'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod H^1_s(F_v,T') & \longrightarrow & \prod H^1_s(F_v,T) & \longrightarrow & \prod H^1_s(F_v,T'')
\end{array}$$

Here all columns are exact, as are the bottom two rows (using Lemma I.3.1 for the singular row). The desired maps and exactness follow from an easy diagram chase. ☐

The map $H^1_f(F,T) \to H^1_f(F,T'')$ need not be surjective in general, although as we will see later one can often force surjectivity by modifying the finite/singular structures. As a consequence of Lemma 3.1, we have the following useful result.

LEMMA 3.2. *Suppose that $T$ is an $l$-adic $G_F$-module over $A$ and that $\alpha \in A$ is such that $(\alpha T)^{G_F} = (T/\alpha T)^{G_F} = 0$. Then $H^1_f(F,T[\alpha])$ injects into $H^1_f(F,T)$, and under this identification it identifies with $H^1_f(F,T)[\alpha]$.*

PROOF. Consider the exact sequences

$$0 \to T[\alpha] \to T \xrightarrow{\alpha} \alpha T \to 0$$

and

$$0 \to \alpha T \to T \to T/\alpha T \to 0.$$

By Lemma 3.1 the row and column in the commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & (T/\alpha T)^{G_F} \\
 & & & & & & \downarrow \\
(\alpha T)^{G_F} & \longrightarrow & H^1_f(F,T[\alpha]) & \longrightarrow & H^1_f(F,T) & \longrightarrow & H^1_f(F,\alpha T) \\
 & & & & & \searrow^{\alpha} & \downarrow \\
 & & & & & & H^1_f(F,T)
\end{array}$$

are exact. It follows that if $(\alpha T)^{G_F} = 0$, then $H^1_f(F,T[\alpha])$ injects into $H^1_f(F,T)$. If $(T/\alpha T)^{G_F} = 0$ as well, then $H^1_f(F,\alpha T)$ injects into $H^1_f(F,T)$, and the rest of the lemma follows. ☐

## 4. A finiteness theorem for Selmer groups

Let $\Sigma$ be a finite subset of $M_F$. We define the *weak $\Sigma$-finite/singular structure* $\mathcal{S}_\Sigma$ on $T$ to be the finite/singular structure on $T$ which is unramified away from $\Sigma$ and weak at $\Sigma$. Note that $H^1_{f,\mathcal{S}_\Sigma}(F,T)$ is simply the set of cohomology classes which are unramified at all places $v \notin \Sigma$ but are unrestricted for $v \in \Sigma$. In particular, if $\mathcal{S}$ is any other finite/singular structure on $T$ which is unramified away from $\Sigma$, then $H^1_{f,\mathcal{S}}(F,T) \subseteq H^1_{f,\mathcal{S}_\Sigma}(F,T)$.

We have the following cohomological interpretation of Selmer groups for the weak $\Sigma$-finite/singular structure.

LEMMA 4.1. *Let $T$ be an $l$-adic $G_F$-module over $A$ and let $\Sigma$ be a finite set of places of $F$ sufficiently large for $T$. Then*

$$H^1_{f,\mathcal{S}_\Sigma}(F,T) \cong H^1(F_\Sigma/F,T)$$

*where $F_\Sigma$ is the maximal extension of $F$ unramified outside of $\Sigma$.*

PROOF. See [**Was97**, Proposition 6]. ☐

The interpretation of the weak structure yields the following fundamental finiteness result, which is really just a slight generalization of the weak Mordell-Weil theorem.

Proposition 4.2. *Let $T$ be a finite $l$-adic $G_F$-module over $A$. Then the Selmer group $H^1_{f,\mathcal{S}}(F,T)$ is finite for any finite/singular structure $\mathcal{S}$.*

Proof. Let $\Sigma$ be a finite set of places of $F$ which is sufficiently large for $T$ and such that $\mathcal{S}$ is unramified away from $\Sigma$. Since $H^1_{f,\mathcal{S}}(F,T) \subseteq H^1_{f,\mathcal{S}_\Sigma}(F,T)$, it is enough to show that $H^1_{f,\mathcal{S}_\Sigma}(F,T)$ is finite.

Since $T$ is finite we can choose a finite Galois extension $F'$ of $F$ such that $G_{F'}$ acts trivially on $T$. Enlarge $\Sigma$ to contain all places of $F$ which are ramified in $F'/F$, and let $\Sigma'$ be the set of places of $F'$ lying above places of $\Sigma$. One sees easily that the finite/singular structures $\mathcal{S}_\Sigma$ and $\mathcal{S}_{\Sigma'}$ are compatible in the sense that there is a commutative diagram

$$\begin{array}{ccc} H^1(F,T) & \longrightarrow & \oplus_v H^1_{s,\mathcal{S}_\Sigma}(F_v,T) \\ \downarrow & & \downarrow \\ H^1(F',T) & \longrightarrow & \oplus_{v'} H^1_{s,\mathcal{S}_{\Sigma'}}(F'_{v'},T) \end{array}$$

We now get an induced map on the kernels of the horizontal maps, which are just the Selmer groups:

$$H_{f,\mathcal{S}_\Sigma}(F,T) \to H_{f,S'_\Sigma}(F',T).$$

Let ker be the kernel of this map; it sits in an exact commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker & \longrightarrow & H^1(F'/F,T) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1_{f,\mathcal{S}_\Sigma}(F,T) & \longrightarrow & H^1(F,T) & \longrightarrow & \oplus_v H^1_{s,\mathcal{S}_\Sigma}(F_v,T) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{f,\mathcal{S}_{\Sigma'}}(F',T) & \longrightarrow & H^1(F',T) & \longrightarrow & \oplus_{v'} H^1_{s,\mathcal{S}_{\Sigma'}}(F'_{v'},T) \end{array}$$

It is clear from the cocycle description that $H^1(F'/F,T)$ is finite; thus ker is finite. Since $G_{F'}$ acts trivially on $T$, it follows immediately from Lemma 4.1 that $H^1_{f,\mathcal{S}_{\Sigma'}}(F',T)$ identifies with $\mathrm{Hom}(\mathrm{Gal}(F''/F'),T)$, where $F''$ is the maximal abelian extension of $F$ unramified outside of $\Sigma$ and of exponent $\#T$. By [**Sil86**, Chapter 8, Proposition 1.6], $F''/F'$ is a finite extension, so this Hom-group is finite. The proposition follows, as $H^1_{f,\mathcal{S}_\Sigma}(F,T)$ lies between two finite groups. □

We have the following version of this result when $T$ is infinite.

Proposition 4.3. *Let $T$ be a finitely generated (resp. discrete) $l$-adic $G_F$-module over $A$. Then the Selmer group $H^1_{f,\mathcal{S}}(F,T)$ is finitely generated (resp. of finite corank) over $\mathbf{Z}_l$ for any finite/singular structure $\mathcal{S}$.*

Proof. Let $\Sigma$ be a finite set of places of $F$ which is sufficiently large for $T$ and such that $\mathcal{S}$ is unramified away from $\Sigma$. Since $\mathbf{Z}_l$ is noetherian it suffices to show that $H^1(F_\Sigma/F,T)$ is finitely generated (resp. of finite corank). Let $G$ denote the Galois group of $F_\Sigma/F$. By [**Sil86**, Chapter 8, Proposition 1.6], the quotient of $G$

by $G^m$ is finite for every $m$. Since $G$ is profinite, this implies that $G$ is topologically finitely generated. The proposition now follows from the fact that the $\mathbf{Z}_l$-module of continuous maps from $G$ to $T$ is finitely-generated (resp. of finite corank). See [**Rub00**, Appendix B, Proposition 1.7] for a slightly more general statement. □

## 5. The Kolyvagin pairing

In this section we will compile the Tate local dualities over all places of $F$ to define a global pairing which will be of fundamental importance in our annihilation theorems for Selmer groups.

Let $T$ be an $l$-adic $G_F$-module over $A$ and assume that we are given a finite/singular structure $\mathcal{S}$ on $T$. Let $\mathcal{S}^*$ be the Cartier dual finite/singular structure on $T^*$; we omit both of these structures from our notation for the remainder of the section. For every place $v$ of $F$, let $\langle \cdot, \cdot \rangle_v$ denote the perfect Tate local pairing

$$H^1_s(F_v, T) \otimes_{\mathbf{Z}_l} H^1_f(F_v, T^*) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

We define the *Kolyvagin pairing*

$$\langle \cdot, \cdot \rangle : \left( \bigoplus_{v \in M_F} H^1_s(F_v, T) \right) \otimes_{\mathbf{Z}_l} H^1_f(F, T^*) \to \mathbf{Q}_l/\mathbf{Z}_l$$

as follows:

$$\langle (c_v), d \rangle = \sum_{v \in M_F} \langle c_v, d_v \rangle_v \, .$$

That this is well-defined is immediate from the fact that $(c_v)$ is an element of the direct sum and thus zero for almost all $v$.

In order to prove our main theorem regarding this global pairing we will need the following standard result on the Brauer group of a global field. We include a cohomological proof for lack of an adequate reference.

LEMMA 5.1. *For any $c \in H^2(F, F_{\mathrm{s}}^\times)$, the restriction $c_v \in H^2(F_v, F_{v,\mathrm{s}}^\times)$ vanishes for almost all $v$.*

PROOF. Since $F_{\mathrm{s}}^\times$ is a discrete $G_F$-module, there is some finite Galois extension $F'/F$ such that $c$ lies in $H^2(F'/F, F'^\times)$. Let $\tilde{c} : \mathrm{Gal}(F'/F) \times \mathrm{Gal}(F'/F) \to F'^\times$ be some choice of cocycle representing $c$. $\mathrm{Gal}(F'/F)$ is finite, so $\tilde{c}$ takes on only finitely many values. Let $\Sigma$ be the subset of $M_F$ of all archimedean places, all places where $F'/F$ is ramified and all places at which elements of $F'^\times$ in the image of $\tilde{c}$ have non-trivial valuation; $\Sigma$ is finite.

Fix $v \notin \Sigma$; we will show that $c_v = 0$. Let $v'$ be the place of $F'$ lying over $v$, via our fixed embedding $F_{\mathrm{s}} \hookrightarrow F_{v,\mathrm{s}}$. Since the image of $\tilde{c}$ has trivial $v$-adic valuation, the cohomology class $c_v$ lies in the image of the natural map

$$H^2\left(F'_{v'}/F_v, \mathcal{O}^\times_{F'_{v'}}\right) \to H^2(F'_{v'}/F_v, {F'_{v'}}^\times).$$

We will show that the source of this map vanishes.

Consider the exact sequence of $\mathrm{Gal}(F'_{v'}/F_v)$-modules

$$0 \to U_1 \to \mathcal{O}^\times_{F'_{v'}} \to k^\times \to 0$$

where $U_1$ is the group of units of $\mathcal{O}_{F'_{v'}}$ congruent to 1 modulo the maximal ideal and $k$ is the residue field of $\mathcal{O}_{F'_{v'}}$. The long exact sequence in cohomology together with the fact that $H^i(F'_{v'}/F_v, U_1) = 0$ for $i \geq 1$ (see [**Ser79**, Chapter 12, Section 3, Lemma 2]) shows that

$$H^2\left(F'_{v'}/F_v, \mathcal{O}^\times_{F'_{v'}}\right) \cong H^2(F'_{v'}/F_v, k^\times).$$

Since $F'_{v'}/F_v$ is unramified, the computation of the cohomology of a finite cyclic group (see [**Ser79**, Chapter 8, Section 4]) and the fact that the norm is surjective on a finite field shows that this last cohomology group is trivial. This completes the proof. □

For a proof of Lemma 5.1 using the cohomology of the ideles, see [**?**, Section 7, Proposition 7.3 and Section 9.6]. (Note that Tate doesn't actually prove that the maps he is considering are the restriction maps, but it is not difficult to check this.) For a proof in terms of division algebras, see [**Pie82**, Chapter 18, Section 5].

We are now in a position to prove the following consequence of global class field theory. For any $l$-adic $G_F$-module $T$, consider the map

$$H^1(F, T) \to \prod_v H^1_s(F_v, T). \tag{5.1}$$

We define the *compactly supported cohomology* $H^1_c(F, T)$ to be the $A$-submodule of $H^1(F, T)$ which has image under (5.1) in the direct sum rather than the direct product:

$$H^1_c(F, T) \to \bigoplus_v H^1_s(F_v, T). \tag{5.2}$$

That is, $H^1_c(F, T)$ consists of those global cohomology classes which are locally unramified almost everywhere. Note that $H^1_c(F, T) = H^1(F, T)$ by Lemma 1.2 if $T$ is discrete.

Proposition 5.2. *Let $T$ be an $l$-adic $G_F$-module over $A$. Then the image of (5.2) is orthogonal to all of $H^1_f(F, T^*)$ under the Kolyvagin pairing.*

Proof. Consider first the commutative diagram

$$\begin{array}{ccc}
H^1(F,T) \otimes_{\mathbf{Z}_l} H^1(F,T^*) & \longrightarrow & \prod_v H^1(F_v,T) \otimes_{\mathbf{Z}_l} H^1(F_v,T^*) \\
\downarrow & & \downarrow \\
H^2(F, T \otimes_{\mathbf{Z}_l} T^*) & \longrightarrow & \prod_v H^2(F_v, T \otimes_{\mathbf{Z}_l} T^*) \\
\downarrow & & \downarrow \\
H^2\big(F, \mu_{l^\infty}(F_s)\big) & \longrightarrow & \prod_v H^2\big(F_v, \mu_{l^\infty}(F_{v,s})\big) \\
\downarrow & & \downarrow \cong \\
H^2(F, F_s^\times) & \longrightarrow & \prod_v H^2(F_v, F_{v,s}^\times)
\end{array}$$

Here all horizontal maps are restriction maps and the vertical maps are cup product, Cartier duality and the map on cohomology coming from the inclusion of the $l$-power roots of unity into the multiplicative group. It follows from the commutativity of this that the diagram

$$\begin{array}{ccc} H^1_c(F,T)\otimes_{\mathbf{Z}_l} H^1_f(F,T^*) & \longrightarrow & \oplus_v H^1_s(F_v,T)\otimes_{\mathbf{Z}_l} H^1_f(F_v,T^*) \\ \downarrow & & \downarrow \\ H^2(F,F_{\mathrm{s}}^\times) & \longrightarrow & \oplus_v H^2(F_v,F_{v,\mathrm{s}}^\times) \end{array} \tag{5.3}$$

is commutative as well. Here we are using Lemma 5.1 to insure that the bottom map is well-defined.

By [**?**, Section 9.6] and [**Mil86**, Appendix A, Theorem 7], there is an exact sequence

$$H^2(F,F_{\mathrm{s}}^\times)\to \bigoplus_v H^2(F_v,F_{v,\mathrm{s}}^\times)\to \mathbf{Q}/\mathbf{Z}\to 0 \tag{5.4}$$

where the last map is the summation map; here $H^2(F_v,F_{v,\mathrm{s}}^\times)$ is identified via local class field theory with $\mathbf{Q}/\mathbf{Z}$, $\frac{1}{2}\mathbf{Z}/\mathbf{Z}$ or 0 in the usual way.

Fix $c\in H^1_c(F,T)$ and $d\in H^1_f(F,T^*)$. Following $c\otimes d$ clockwise around (5.3) and then mapping it by summation to $\mathbf{Q}/\mathbf{Z}$ yields the global pairing $\langle (c_{v,s}),d\rangle$ by definition. Following $c\otimes d$ around (5.3) in the counter-clockwise direction shows that it maps to $\mathbf{Q}/\mathbf{Z}$ via $H^2(F,F_{\mathrm{s}}^\times)$; by (5.4) this map vanishes, which completes the proof. ☐

## 6. Shafarevich-Tate groups

To define a pairing between Selmer groups we will need some basic facts on Shafarevich-Tate groups. If $\Sigma$ is any set of places of $F$, let $F_\Sigma$ be the maximal extension of $F$ unramified away from $\Sigma$.

Definition 6.1. Let $T$ be an $l$-adic $G_F$-module over $A$ and let $\Sigma$ be an arbitrary set of places of $F$. The *first* $\Sigma$*-Shafarevich-Tate group* of $T$ is

$$Ш^1_\Sigma(F,T)=\ker\left(H^1\big(F_\Sigma/F,T^{G_{F_\Sigma}}\big)\to \bigoplus_{v\in\Sigma}H^1(F_v,T)\right).$$

By [**Mil86**, Chapter 1, Theorem 4.10(a)], $Ш^1_\Sigma(F,T)$ is finite for any set of places $\Sigma$ and any finite $l$-adic $G_F$-module $T$.

If $\Sigma$ is sufficiently large for $T$, then by Lemma 4.1 the inflation map

$$H^1(F_\Sigma/F,T)\hookrightarrow H^1(F,T)$$

identifies $H^1(F_\Sigma/F,T)$ with $H_{f,\mathcal{S}_\Sigma}(F,T)$; thus we can also write

$$\begin{aligned} Ш^1_\Sigma(F,T) &= \ker\left(H^1_{f,\mathcal{S}}(F,T)\to \bigoplus_{v\in\Sigma}H^1(F_v,T)\right)\\ &= \ker\left(H^1(F,T)\to \bigoplus_{v\in\Sigma}H^1(F_v,T)\times \bigoplus_{v\notin\Sigma}H^1_{s,\Sigma_\Sigma}(F_v,T)\right). \end{aligned} \tag{6.1}$$

That is, a cohomology class lies in $Ш^1_\Sigma(F,T)$ if and only if it is unramified away from $\Sigma$ and is actually zero at all places of $\Sigma$. If $\Sigma\subseteq\Sigma'$ are sufficiently large for $T$, then by (6.1) there is a canonical inclusion

$$Ш^1_{\Sigma'}(F,T)\hookrightarrow Ш^1_\Sigma(F,T).$$

Write $Ш^1(F,T)$ for $Ш^1_{M_F}(F,T)$. We will need the following lemma.

LEMMA 6.2. *Let $T$ be a finite $l$-adic $G_F$-module over $A$ and suppose that*

$$Ш^1(F,T) = 0.$$

*Then there is some finite set of places $\Sigma$ such that $Ш^1_\Sigma(F,T) = 0$.*

PROOF. Choose a finite set of places $\Sigma_0$ which is sufficiently large for $T$. By our remarks above, $Ш^1_{\Sigma_0}(F,T)$ is finite. Since $Ш^1(F,T) = 0$, for every $x \in Ш^1_{\Sigma_0}(F,T)$ there is some place $v_x$ such that $x_{v_x} \neq 0$. Taking $\Sigma$ to contain $\Sigma_0$ and all of the places $v_x$ and using (6.1) proves the lemma. □

DEFINITION 6.3. Let $T$ be an $l$-adic $G_F$-module over $A$ and let $\Sigma$ be an arbitrary set of places of $F$. The *second $\Sigma$-Shafarevich-Tate group* of $T$ is

$$Ш^2_\Sigma(F,T) = \ker\left(H^2\big(F_\Sigma/F, T^{G_{F_\Sigma}}\big) \to \bigoplus_{v\in\Sigma} H^2(F_v,T)\right).$$

We have the following fundamental relationship between $Ш^1$ and $Ш^2$.

PROPOSITION 6.4. *Let $T$ be a finite $l$-adic $G_F$-module over $A$ and let $\Sigma$ be a finite set of places containing $\Sigma_l$. Then there is a perfect pairing*

$$Ш^1_\Sigma(F,T) \oplus Ш^2_\Sigma(F,T^*) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

PROOF. See [**Mil86**, Chapter 1, Theorem 4.10]. □

Consider now a surjection $T \twoheadrightarrow T''$ of $l$-adic $G_F$-modules over $A$. Assume that $T''$ is given a finite/singular structure induced by one on $T$. In general there is no reason to expect the induced map on Selmer groups

$$H^1_f(F,T) \to H^1_f(F,T'')$$

to be surjective. However, if a certain Shafarevich-Tate group vanishes, we can obtain a partial result.

LEMMA 6.5. *Let $0 \to T' \to T \to T'' \to 0$ be an exact sequence of finite $l$-adic $G_F$-modules over $A$. Assume that $T$ has a finite/singular structure $\mathcal{S}$ and let $T'$ and $T''$ have the induced finite/singular structures. Suppose that $Ш^1(F,T'^*) = 0$. Then for any $x'' \in H^1_{f,\mathcal{S}}(F,T'')$, there is a finite set of places $\Sigma$ and an element $x \in H^1(F,T)$ such that*

- *$x$ maps to $x''$ under the map $H^1(F,T) \to H^1(F,T'')$;*
- *$x_v \in H^1_{f,\mathcal{S}}(F_v,T)$ for all $v \notin \Sigma$.*

PROOF. Let $\Sigma$ be a finite set of places which is sufficiently large for $T$ and such that $Ш^1_\Sigma(F,T'^*) = 0$. By Proposition 6.4 this implies that $Ш^2_\Sigma(F,T') = 0$. Let $\mathcal{S}'$ be the finite/singular structure on $T$ which agrees with $\mathcal{S}$ away from $\Sigma$ and which has the weak structure at all places of $\Sigma$. We will also write $\mathcal{S}'$ for the induced finite/singular structures on $T'$ and $T''$. The long exact sequence in $G_F$-cohomology and Lemma 4.1 yield an exact sequence

$$H^1_{f,\mathcal{S}'}(F,T) \to H^1_{f,\mathcal{S}'}(F,T'') \to H^2(F_\Sigma/F,T'). \tag{6.2}$$

Since $x'' \in H^1_{f,\mathcal{S}}(F,T'')$, its restriction to $H^1(F_v,T'')$ lies in $H^1_{f,\mathcal{S}}(F_v,T'')$ for all places $v$. For $v \in \Sigma$, consider the image of $x''$ under the natural map

$$H^1_{f,\mathcal{S}'}(F,T'') \to H^2(F_\Sigma/F,T') \to H^2(F_v,T').$$

By Lemma I.3.1, $H^1_{f,\mathcal{S}}(F_v, T'')$ is annihilated by the boundary map

$$H^1(F_v, T'') \to H^2(F_v, T'),$$

so $x''$ maps to 0 in $H^2(F_v, T')$. This shows that $x''$ maps into $Ш^2_\Sigma(F, T') \subseteq H^2(F_\Sigma/F, T')$. But this group vanishes, so by the exactness of (6.2) $x''$ pulls back to an element $x$ of $H^1_{f,\mathcal{S}'}(F, T)$. This $x$ satisfies the required conditions. □

## 7. The Bockstein pairing

Let

$$0 \to T' \xrightarrow{\alpha} T \xrightarrow{\beta} T'' \to 0$$

be an exact sequence of finite $l$-adic $G_F$-modules over $A$. Let $T$ have a fixed finite/singular structure and let $T'$ and $T''$ have the induced finite/singular structures. Assume also that $Ш^1(F, T'^*)$ vanishes. Under these hypotheses we will define the *Bockstein pairing*

$$\{\cdot,\cdot\}_{\alpha,\beta} : H^1_f(F, T'') \otimes H^1_f(F, T'^*) \to \mathbf{Q}_l/\mathbf{Z}_l$$

which we will study in much more detail later.

Fix $x'' \in H^1_f(F, T'')$ and $y' \in H^1_f(F, T'^*)$. Since we assumed that $Ш^1(F, T'^*) = 0$, by Lemma 6.5 there is a finite set $\Sigma$ of places of $F$ and an element $x \in H^1(F, T)$ such that $x_{v,s} = 0$ for $v \notin \Sigma$ and such that $x$ maps to $x''$ under the map $H^1(F, T) \to H^1(F, T'')$. For each $v \in \Sigma$, consider the diagram

(7.1)

$$\begin{array}{ccccccccc}
 & & & & x & \longmapsto & x'' & & \\
 & & & & H^1(F,T) & \longrightarrow & H^1(F,T'') & & x'' \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_s(F_v,T') & \longrightarrow & H^1_s(F_v,T) & \longrightarrow & H^1_s(F_v,T'') & & 0 \\
 & & x'_v & \longmapsto & x_{v,s} & \longmapsto & 0 & &
\end{array}$$

Here the bottom row is the exact sequence of Lemma I.3.1. Since $x''_v$ lies in $H^1_f(F_v, T'')$ and $x$ maps to $x''$, the image of $x_{v,s}$ in $H^1_s(F_v, T'')$ is zero; thus by (7.1) there is an element $x'_v \in H^1_s(F_v, T')$ which maps to $x_{v,s}$. Set $x'_v = 0$ for $v \notin \Sigma$. We define

$$\{x'', y'\}_{\alpha,\beta} = \langle (x'_v), y' \rangle \in \mathbf{Q}_l/\mathbf{Z}_l.$$

We must check that this definition is independent of the choice of places $\Sigma$ and the choice of lifting $x$. Since for any fixed lifting $x$ we can always freely enlarge $\Sigma$, it is enough to check this for two different liftings of $x''$ for the same $\Sigma$. However, the difference of these liftings lies in $H^1(F, T')$, and Proposition 5.2 now gives the desired independence.

The Bockstein pairing can be defined without the assumption on $Ш^1(F, T'^*)$; see [**FPR94**, Chapitre 2, Section 1.4].

CHAPTER 3

# Annihilation theorems for Selmer groups

We now turn to the definition of partial geometric Euler systems and the corresponding annihilation theorems for Selmer groups. The same methods will also yield a non-degeneracy result for the Bockstein pairing.

## 1. Partial geometric Euler systems

Let $A$ and $F$ be as in Chapter II. Let $T$ be a finitely generated $l$-adic $G_F$-module over $A$ with a finite/singular structure $\mathcal{S}$. If $C$ is an $A$-submodule of $H^1(F,T)$ and $v$ is a place of $F$, we write $C_{v,s}$ for the image of $C$ in $H^1_s(F_v,T)$.

Definition 1.1. Let $\mathcal{L}$ be a (possibly infinite) set of places of $F$ and let $\eta$ be an ideal of $A$. A *partial (geometric) Euler system* $\{C^v\}_{v\in\mathcal{L}}$ of depth $\eta$ for $T$ (with the structure $\mathcal{S}$) at $\mathcal{L}$ is an assignment of $A$-submodules $C^v \subseteq H^1(F,T)$ for each $v \in \mathcal{L}$ such that

- $C^v_{w,s} = 0$ for all places $w \neq v$;
- $H^1_s(F_v,T)/C^v_{v,s}$ is killed by $\eta$.

If in addition $C^v$ vanishes in $H^1_s(F_v,T/\eta T)$ for all $v \in \mathcal{L}$, we say that the partial Euler system has *strict depth of* $\eta$. In this case the image of each $C^v$ in $H^1(F,T/\eta T)$ lies in $H^1_f(F,T/\eta T)$, and we define the *Euler module* $\Phi$ of $\{C^v\}_{v\in\mathcal{L}}$ to be the $A$-submodule of $H^1_f(F,T/\eta T)$ generated by the image of $C^v$ for all $v \in \mathcal{L}$.

The next result explains how partial Euler systems behave under pushforward.

Lemma 1.2. *Let $j : T \twoheadrightarrow T''$ be a surjection of $l$-adic $G_F$-modules over $A$ and let $T''$ have the finite/singular structure induced by $\mathcal{S}$. Assume that $T$ admits a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth $\eta$. Let $\mathfrak{d}$ be an ideal of $A$ which annihilates the cokernel of*

$$H^1_{s,\mathcal{S}}(F_v,T) \to H^1_{s,\mathcal{S}}(F_v,T'')$$

*for every $v \in \mathcal{L}$. Then $\{j_*C^v\}_{v\in L}$ is a partial Euler system for $T''$ of depth $\eta\mathfrak{d}$.*

Proof. That $j_*C^v$ is supported only at $v$ is immediate from the definition of the induced finite/singular structure on $T''$. The assertions at $v$ are clear from the definitions. □

The following result is the first step in the proof of our annihilation theorems for Selmer groups. For a set of places $\mathcal{L}$, define

$$H^1_{\mathcal{L}}(F,T^*) = \ker\left(H^1(F,T^*) \to \prod_v H^1(F_v,T^*)\right).$$

Note that $Ш^1_{\mathcal{L}}(F,T^*) \subseteq H^1_{\mathcal{L}}(F,T^*)$.

LEMMA 1.3. *Let $T$ be a finitely generated $l$-adic $G_F$-module over $A$ with a finite/singular structure $\mathcal{S}$. Suppose that $T$ admits a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth $\eta$. Then*

$$\eta H^1_f(F,T^*) \subseteq H^1_{\mathcal{L}}(F,T^*).$$

PROOF. Fix $v \in \mathcal{L}$. By the definition of a partial Euler system, $C^v$ maps to 0 in every singular cohomology group except for $H^1_s(F_v,T)$. In particular, $C^v \subseteq H^1_c(F,T)$. In addition, for $c \in C^v$ and $d \in H^1_f(F,T^*)$, the Kolyvagin pairing $\langle (c_w), d\rangle$ is simply the Tate local pairing at $v$:

$$\langle (c_w), d\rangle = \langle c_v, d_v\rangle_v$$

Proposition II.5.2 now shows that $C^v_{v,s}$ and the image of $H^1_f(F,T^*) \to H^1_f(F_v,T^*)$ are orthogonal under the Tate local pairing at $v$. Since this is a perfect pairing and $\eta$ kills $H^1_s(F_v,T)/C^v_{v,s}$, it follows immediately that $\eta$ kills the image of $H^1_f(F,T^*)$ in $H^1_f(F_v,T^*)$. This is the statement of the lemma. □

Note that we can not conclude from Lemma 1.3 that $\eta H^1_f(F,T^*)$ is contained in $Ш^1_{\mathcal{L}}(F,T^*)$ because of possible bad behavior at the bad places for $T^*$.

## 2. The key lemmas

In this section we prove the key lemmas for the annihilation theorems for Selmer groups. Let $T$ be a finite $l$-adic $G_F$-module over $A$ with a finite/singular structure $\mathcal{S}$. Let $F' = F(T)$ be the splitting field of $T$; it is a finite Galois extension of $F$ and we set $\Delta = \operatorname{Gal}(F'/F)$. Note that $\Delta$ injects into $\operatorname{Aut}_A T$.

If $\tau$ is any element of $\Delta$, we define $\mathcal{L}_\tau$ to be the set of non-archimedean places of $F$ which are unramified in the extension $F'/F$ and which have Frobenius conjugate to $\tau$ over $F'$. That is,

$$\begin{aligned}\mathcal{L}_\tau =\{&v \in M_F \mid v \text{ non-archimedean}, F'_v/F_v \text{ unramified, there} \\ &\text{exists } v' \in M_{F'} \text{ such that } v'|v \text{ and } \operatorname{Fr}_{F'/F}(v') = \tau\}.\end{aligned}$$

By the Tchebatorev density theorem $\mathcal{L}_\tau$ has positive density in $M_F$.

We also make the analogous definitions for $T[\mathfrak{m}]$: let $F_\mathfrak{m} = F(T[\mathfrak{m}])$ be its splitting field and set $\Delta_\mathfrak{m} = \operatorname{Gal}(F_\mathfrak{m}/F)$; it injects into $\operatorname{Aut}_k(T[\mathfrak{m}])$.

We will say that an element of $\Delta_\mathfrak{m}$ or $\Delta$ is a *non-scalar involution* if it acts in that way (as an $A$-linear endomorphism) on $T[\mathfrak{m}]$ or $T$, respectively. The next lemma shows that it is easy to lift non-scalar involutions.

LEMMA 2.1. *Assume $l \neq 2$. Suppose that there is a non-scalar involution $\tau_\mathfrak{m}$ in $\Delta_\mathfrak{m}$. Then there exists a non-scalar involution $\tau$ in $\Delta$ lifting $\tau_\mathfrak{m}$.*

PROOF. Note that the map $\operatorname{Aut}_A T \to \operatorname{Aut}_k(T[\mathfrak{m}])$ is surjective with kernel an $l$-group. Since $\Delta$ and $\Delta_\mathfrak{m}$ inject into these groups and $\Delta$ surjects onto $\Delta_\mathfrak{m}$, we see that we can lift $\tau_\mathfrak{m}$ to some element $\tau_0$ in $\Delta$ which has order 2 times a power of $l$. Taking an appropriate $l$-power of $\tau_0$ we obtain an involution $\tau \in \Delta$. Since $l \neq 2$, $\tau$ will still reduce to $\tau_\mathfrak{m}$ in $\Delta_\mathfrak{m}$ and thus is still non-scalar □

LEMMA 2.2. *Let $T$ be a finite $l$-adic $G_F$-module over $A$. Suppose that the following conditions hold:*

- *$l \neq 2$;*
- *$T[\mathfrak{m}]$ is absolutely irreducible as a $G_F$-module over $k$;*
- *There is a non-scalar involution $\tau_\mathfrak{m} \in \Delta_\mathfrak{m}$.*

*Let $\tau \in \Delta$ be some non-scalar involution lifting $\tau_{\mathfrak{m}}$. Let $\mathcal{L}$ be a set of places cofinite in $\mathcal{L}_\tau$. Then $H^1_{\mathcal{L}}(F,T) \subseteq H^1(\Delta,T)$; here we regard $H^1(\Delta,T)$ as a subgroup of $H^1(F,T)$ via inflation. In particular, $\text{Ш}^1_{\mathcal{L}}(F,T) \subseteq H^1(\Delta,T)$.*

Proof. Consider the exact sequence

$$1 \to G_{F'} \to G_F \to \Delta \to 1.$$

This yields an inflation-restriction exact sequence

$$0 \to H^1(\Delta,T) \to H^1(F,T) \to \mathrm{Hom}_\Delta(G_{F'}^{\mathrm{ab}},T). \tag{2.1}$$

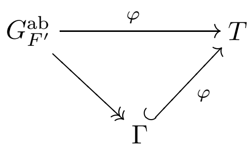
Chasing through the definitions, one finds that $\delta \in \Delta$ acts on $g \in G_{F'}^{\mathrm{ab}}$ as follows: let $\tilde{\delta}$ be any lifting of $\delta$ to $G_F$, and set ${}^{\delta}g = \tilde{\delta} g \tilde{\delta}^{-1}$. One checks immediately that this is a well-defined action and yields an element of $G_{F'}^{\mathrm{ab}}$. Homomorphisms in $\mathrm{Hom}_\Delta(G_{F'}^{\mathrm{ab}},T)$ are equivariant for this action of $\Delta$ on $G_{F'}^{\mathrm{ab}}$ and the natural action of $\Delta$ on $T$.

Let $c \in H^1_{\mathcal{L}}(F,T)$ satisfy $c_v = 0$ for all $v \in \mathcal{L}$. Let $\varphi : G_{F'}^{\mathrm{ab}} \to T$ be the image of $c$ in $\mathrm{Hom}_\Delta(G_{F'}^{\mathrm{ab}},T)$. To prove the lemma we must show that $\varphi = 0$.

Let $F''$ be the fixed field of the kernel of $\varphi$ and set $\Gamma = \mathrm{Gal}(F''/F')$; we have an exact sequence

$$1 \to \Gamma \to \mathrm{Gal}(F''/F) \to \Delta \to 1$$

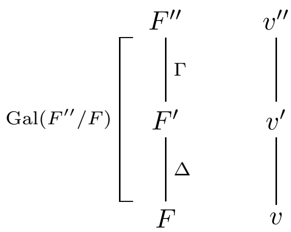
and a commutative diagram




In particular, $\Gamma$ is a finite abelian $l$-group since it injects into $T$.

Let $\tau$ be our fixed non-scalar involution in $\Delta$. Since $\Gamma$ has odd order we can lift $\tau$ to an involution $\tilde{\tau}$ in $\mathrm{Gal}(F''/F)$ as in the proof of Lemma 2.1. Let $g$ be any element of $\Gamma$ and consider $\tilde{\tau} g \in \mathrm{Gal}(F''/F)$. By the Tchebatorev density theorem there exists an unramified place $v''$ of $F''$ such that $\mathrm{Fr}_{F''/F}(v'') = \tilde{\tau} g$. Setting $v' = v''|_{F'}$ and $v = v''|_F$, we have the following situation:

(2.2)




By standard properties of Frobenius elements we have

$$\mathrm{Fr}_{F'/F}(v') = \mathrm{Fr}_{F''/F}(v'')|_{F'} = \tilde{\tau} g|_{F'} = \tau$$

(so in particular $v \in \mathcal{L}_\tau$) and

$$\mathrm{Fr}_{F''/F'}(v'') = \mathrm{Fr}_{F''/F}(v'')^{\deg(v'/v)} = (\tilde{\tau} g)^2. \tag{2.3}$$

Here by $\deg(v'/v)$ we mean the degree of the local field extension $F'_{v'}/F_v$; it is 2 since $\mathrm{Fr}_{F'/F}(v') = \tau$ has order 2 and $v'/v$ is unramified. Note also that by the Tchebatorev density theorem there are infinitely many possible choices of such $v''$;

in particular, we can avoid any finite set of places of $\mathcal{L}_\tau$ and therefore assume that $v \in \mathcal{L}$. Thus, by hypothesis, $c_v = 0$.

We claim that since $c_v = 0$ we have $\varphi(\mathrm{Fr}_{F''/F'}(v'')) = 0$. To see this begin with the commutative diagram

$$\begin{array}{ccc} c \in H^1(F,T) & \xrightarrow{\mathrm{res}_v} & H^1(F_v,T) \\ \downarrow & & \downarrow \\ \varphi \in \mathrm{Hom}(G_{F'},T) & \longrightarrow & \mathrm{Hom}(G_{F'_{v'}},T) \end{array}$$

Since $\varphi$ factors through $\Gamma = \mathrm{Gal}(F''/F')$, $\varphi|_{G_{F'_{v'}}}$ factors through $\mathrm{Gal}(F''_{v''}/F'_{v'})$, which is generated by $\mathrm{Fr}_{F''/F'}(v'')$. Since $c_v = 0$ we have $\varphi|_{G_{F'_{v'}}} = 0$, which we now see says precisely that $\varphi(\mathrm{Fr}_{F''/F'}(v'')) = 0$, as claimed.

Combining (2.3) with this, we conclude that

$$\varphi(\tilde{\tau} g \tilde{\tau} g) = 0. \tag{2.4}$$

Since $\tilde{\tau}^2 = 1$, we can write $\tilde{\tau} g \tilde{\tau} g = \tilde{\tau} g \tilde{\tau}^{-1} g$; by the definition of the action of $\Delta$ on $G_{F'}^{\mathrm{ab}}$ this is nothing other than ${}^{\tau}g \cdot g$. Since $\varphi$ is $\Delta$-equivariant (2.4) now implies that

$$\tau\varphi(g) = -\varphi(g). \tag{2.5}$$

(2.5) holds for all $g \in \Gamma$, so if we let $\Psi$ be the $A$-submodule of $T$ generated by $\varphi(\Gamma)$, then we have $\Psi \subseteq T^-$. Here by $T^-$ we mean the $-1$ eigenspace for the action of $\tau$ on $T$. Note also that $\Psi$ is stable under the action of $\Delta$, as $\varphi$ is $\Delta$-equivariant and the action of $\Delta$ is $A$-linear.

Consider $\Psi[\mathfrak{m}] \subseteq T^-[\mathfrak{m}] \subseteq T[\mathfrak{m}]$. Since $\tau_{\mathfrak{m}}$ acts as a non-scalar, the second inclusion is strict. As $\Psi[\mathfrak{m}]$ is $\Delta$-stable and $T[\mathfrak{m}]$ is irreducible as a $\Delta$-module, this implies that $\Psi[\mathfrak{m}] = 0$. By Lemma B.7.3 this implies that $\Psi = 0$; thus $\varphi = 0$, which is what we were trying to prove. ☐

Note that it is implicit in the above proof that $T[\mathfrak{m}]$ has dimension at least 2 over $k$, as otherwise all $k$-linear automorphisms of $T[\mathfrak{m}]$ are scalar. To get a result for the one dimensional case one can mimic the above proof with $\tau = 1$; this is a special case of the next result.

We now give an alternate version of Lemma 2.2 which is due to Rubin. In fact, Lemma 2.2 is a special case of this result, but we include it separately as the proof is of independent interest.

LEMMA 2.3. *Let $T$ be a finite $l$-adic $G_F$-module over $A$. Suppose that the following conditions hold:*

- *$T[\mathfrak{m}]$ is absolutely irreducible as a $G_F$-module over $k$;*
- *There is a $\tau_{\mathfrak{m}} \in \Delta_{\mathfrak{m}}$ such that $T[\mathfrak{m}]/(\tau_{\mathfrak{m}} - 1)T[\mathfrak{m}] \neq 0$.*

*Let $\tau$ be any lifting of $\tau_{\mathfrak{m}}$ to $\Delta$. Let $\mathcal{L}$ be a set of places cofinite in $\mathcal{L}_\tau$. Then $H^1_{\mathcal{L}}(F,T) \subseteq H^1(\Delta,T)$. In particular, $Ш^1_{\mathcal{L}}(F,T) \subseteq H^1(\Delta,T)$.*

PROOF. The proof is similar in spirit to that of Lemma 2.2. As before, by (2.1) it is enough to show that for any $c \in H^1(F,T)$ such that $c_v = 0$ for all $v \in \mathcal{L}$, the associated homomorphism $\varphi : G_{F'}^{\mathrm{ab}} \to T$ is trivial. Choose also a representative cocycle $\tilde{c} : G_F \to T$ for the cohomology class $c$. Let $F''$ be some finite extension of $F'$ through which $\tilde{c}$ factors; $\varphi$ necessarily factors through $\mathrm{Gal}(F''/F')$, which we denote by $\Gamma$. That is, we now have maps $\tilde{c} : \mathrm{Gal}(F''/F) \to T$ and $\varphi : \Gamma \to T$, the first a cocycle and the second a homomorphism. Note that $\varphi$ need not be injective on $\Gamma$.

Fix some lifting $\tilde{\tau}$ of $\tau$ to $\mathrm{Gal}(F''/F)$. Choose also some $g \in \Gamma$. By the Tchebatorev density theorem we can find a place $v''$ of $F''$ such that $\mathrm{Fr}_{F''/F}(v'') = \tilde{\tau}g$. Let $v'$ and $v$ be the restriction of $v''$ to $F'$ and $F$ respectively. (This is the same basic set-up as in (2.2).) We have $\mathrm{Fr}_{F'/F}(v') = \tau$, so that $v \in \mathcal{L}_\tau$. As before, we can assume that $v$ avoids any finite set and therefore we can take $v$ to lie in $\mathcal{L}$; thus $c_v = 0$.

$\tilde{c}|_{\mathrm{Gal}(F''_{v''}/F_v)}$ is a coboundary, since $c_v = 0$. (Here we are also using the fact that inflation maps are injective on $H^1$ in order to insure that $\tilde{c}$ really is a coboundary for $\mathrm{Gal}(F''_{v''}/F_v)$.) Thus in particular $\tilde{c}(\mathrm{Fr}_{F''/F}(v'')) \in (\mathrm{Fr}_{F''/F}(v'') - 1)T$; that is,

$$\tilde{c}(\tilde{\tau}g) \in (\tilde{\tau}g - 1)T = (\tau - 1)T. \tag{2.6}$$

Taking $g = 1$ shows that $\tilde{c}(\tilde{\tau}) \in (\tau - 1)T$.

Returning to the case of arbitrary $g \in \Gamma$, by the cocycle relation we have

$$\tilde{c}(\tilde{\tau}g) = \tilde{c}(\tilde{\tau}) + \tilde{\tau}\tilde{c}(g) = \tilde{c}(\tilde{\tau}) + \tau\tilde{c}(g).$$

This lies in $(\tau - 1)T$ by (2.6), so combined with the fact that $\tilde{c}(\tilde{\tau}) \in (\tau - 1)T$, this shows that $\tau\tilde{c}(g)$ lies in $(\tau - 1)T$. Since $(\tau - 1)\tilde{c}(g)$ trivially lies in here, we conclude that

$$\tilde{c}(g) \in (\tau - 1)T$$

for all $g \in \Gamma$.

Thus the image of $\tilde{c}$, and therefore the image of $\varphi$, lies in $(\tau - 1)T$. Letting $\Psi$ be the $A$-submodule of $(\tau - 1)T$ generated by $\varphi(\Gamma)$, the proof continues as before, using the fact that

$$(\tau_{\mathfrak{m}} - 1)T[\mathfrak{m}] \neq T[\mathfrak{m}]$$

and the absolute irreducibility of $T[\mathfrak{m}]$ to show that $\Psi = 0$. □

## 3. The annihilation theorem

The following theorem is essentially due to Flach (see [**Fla92**, Proposition 1.1]), although the ideas go back to Thaine and Kolyvagin and this presentation is due to Mazur. For any $l$-adic $G_F$-module $T$, let $\delta = \delta(T)$ be the $A$-annihilator of the cohomology group $H^1(\Delta, T^*)$.

THEOREM 3.1. *Let $T$ be a finite $l$-adic $G_F$-module over $A$ with a finite/singular structure $\mathcal{S}$. Suppose that one of the following two conditions hold:*

- *$l \neq 2$ and there is a non-scalar involution $\tau_{\mathfrak{m}} \in \Delta_{\mathfrak{m}}$; or*
- *there is a $\tau_{\mathfrak{m}} \in \Delta_{\mathfrak{m}}$ such that $T^*[\mathfrak{m}]/(\tau_{\mathfrak{m}} - 1)T^*[\mathfrak{m}] \neq 0$.*

*Let $\tau$ be an appropriate lifting of $\tau_{\mathfrak{m}}$ to $\Delta$. (That is, $\tau$ is a non-scalar involution in the first case and is an arbitrary lifting in the second case.) Assume also that*

- *$T^*[\mathfrak{m}]$ is absolutely irreducible as a $G_F$-module over $k$;*
- *$T$ admits a partial Euler system $\{C^v\}_{v \in \mathcal{L}}$ of depth $\eta$ for some set of places $\mathcal{L}$ cofinite in $\mathcal{L}_\tau$.*

*Then $\delta\eta$ annihilates the Selmer group $H^1_f(F, T^*)$.*

PROOF. This is immediate from Lemma 1.3 and Lemma 2.2 or Lemma 2.3 (applied to $T^*$), as appropriate. □

We now let $T$ be an arbitrary finitely generated $l$-adic $G_F$-module over $A$. For any ideal $\mathfrak{a}$ of finite index in $A$, $T^*[\mathfrak{a}]$ is finite; we let $F_\mathfrak{a}$ be its splitting field and we set $\Delta_\mathfrak{a} = \mathrm{Gal}(F_\mathfrak{a}/F)$.

Let $\delta = \delta(T)$ be the largest ideal of $A$ which annihilates each of the groups $H^1(\Delta_{l^n}, T^*[l^n])$ for sufficiently large $n$. Let $\mathfrak{d} = \mathfrak{d}(T)$ be the largest ideal of $A$ which annihilates the cokernel of

$$H^1_s(F_v, T) \to H^1_s(F_v, T/l^nT)$$

for $n$ greater than some fixed $n_0$ and all $v \in \mathcal{L}$.

As always, if $l \neq 2$ we can lift a non-scalar involution $\tau_\mathfrak{m} \in \Delta_\mathfrak{m}$ to non-scalar involutions $\tau_n \in \Delta_{l^n}$ for each $n$.

Corollary 3.2. *Let $T$ be a finitely generated $l$-adic $G_F$-module over $A$. Suppose that one of the following two conditions hold:*

- *$l \neq 2$ and there is a non-scalar involution $\tau_\mathfrak{m} \in \Delta_\mathfrak{m}$; or*
- *There is a $\tau_\mathfrak{m} \in \Delta_\mathfrak{m}$ such that $T^*[\mathfrak{m}]/(\tau_\mathfrak{m} - 1)T^*[\mathfrak{m}] \neq 0$.*

*Let $\tau_n$ be appropriate liftings of $\tau_\mathfrak{m}$ to each $\Delta_{l^n}$. If $T^*[\mathfrak{m}]$ has rank one over $k$, assume further that*

- *$T^*/T^*[l^n]$ has no $G_F$-invariants for sufficiently large $n$.*

*Assume also that*

- *$T^*[\mathfrak{m}]$ is absolutely irreducible as a $G_F$-module over $k$;*
- *$T$ admits a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth $\eta$ for some set of places $\mathcal{L}$ cofinite in $\mathcal{L}_{\tau_n}$ for some $n$.*

*Then $\delta\mathfrak{d}\eta$ annihilates the Selmer group $H^1_f(F, T^*)$.*

Proof. Let $c \in H^1_f(F, T^*)$ be any element. Since $T^*$ is discrete, $c$ factors through some finite extension $F'/F$. Let $\tilde{c} : \mathrm{Gal}(F'/F) \to T^*$ be some cocycle representing $c$. $\mathrm{Gal}(F'/F)$ is finite, so $\tilde{c}$ takes on only finitely many values. In particular, its image must lie in $T^*[l^m]$ for some $m$. This means that $c$ lies in the image of the map

$$H^1\big(F, T^*[l^m]\big) \to H^1(F, T^*). \tag{3.1}$$

We can also assume that $m$ is sufficiently large so that

- $T^*/T^*[l^m]$ has no $G_F$-invariants (this is automatic when $T^*[\mathfrak{m}]$ has rank at least two, since $T^*[\mathfrak{m}]$ is absolutely irreducible);
- $T$ admits a partial Euler system of depth $\eta$ for some set of places $\mathcal{L}$ cofinite in $\mathcal{L}_{\tau_m}$;
- $\delta H^1(\Delta_{l^m}, T^*[l^m]) = 0$;
- $\mathfrak{d}$ annihilates the cokernel of $H^1_s(F_v, T) \to H^1_s(F_v, T/l^mT)$ for all $v \in \mathcal{L}$.

It follows from the fact that $(T^*/T^*[l^m])^{G_F} = 0$ and the long exact sequence in cohomology that (3.1) is injective. Since $c$ lies in $H^1_f(F, T^*)$, it follows from our definition of the induced structure that $c$ actually lies in the image of $H^1_f(F, T^*[l^m])$. By Lemma 1.2 the partial Euler system for $T$ induces one of depth $\mathfrak{d}\eta$ for $T/l^mT$, so Theorem 3.1 shows that $\delta\mathfrak{d}\eta$ annihilates $H^1_f(F, T^*[l^m])$; thus it must annihilate $c$ as well. □

## 4. Right non-degeneracy of the Bockstein pairing

Let $T$ be a finitely generated $l$-adic $G_F$-module over $A$. Assume:

- The hypotheses of Corollary 3.2 are satisfied;
- The partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ is of *strict depth* $\eta$;
- $\eta$ is principal, generated by a non-zero divisor (we also write $\eta$ for a fixed generator);
- $H^1(F_\eta/F, T^*[\eta]) = 0$;
- The groups $H^1_f(F_v, T)$ are *divisible by* $\eta$, in the sense that if $\eta c \in H^1_f(F_v, T)$ for some $c \in H^1(F_v, T)$, then $c \in H^1_f(F_v, T)$;

Note that the last hypothesis is satisfied in the case of any of the local finite/singular structures we described in Section I.4, except possibly for the strong structure. Let $\Phi \subseteq H^1_f(F, T/\eta T)$ denote the Euler module of the partial Euler system $\{C^v\}_{v\in\mathcal{L}}$.

By Lemma 2.2 or Lemma 2.3, the assumption that $H^1(F_\eta/F, T^*[\eta]) = 0$ implies that $Ш^1_{\mathcal{L}}(F, T^*[\eta]) = 0$. Thus we can define the Bockstein pairing

$$\{\cdot,\cdot\}_\eta : H^1_f\big(F, T/\eta T\big) \otimes H^1_f(F, T^*[\eta]) \to \mathbf{Q}_l/\mathbf{Z}_l \tag{4.1}$$

associated to the exact sequence

$$0 \to T/\eta T \xrightarrow{\eta} T/\eta^2 T \longrightarrow T/\eta T \to 0;$$

here the first map is multiplication by $\eta$ and the second map is the natural quotient map. (One checks easily that the divisibility hypothesis above insures that the injection $T/\eta T \hookrightarrow T/\eta^2 T$ and the surjection $T/\eta^2 T \twoheadrightarrow T/\eta T$ induce the same finite/singular structure on $T/\eta T$.)

Proposition 4.1. *Let $T$ be as above. The restriction*

$$\{\cdot,\cdot\}_\eta : \Phi \otimes H^1_f\big(F, T^*[\eta]\big) \to \mathbf{Q}_l/\mathbf{Z}_l$$

*of the Bockstein pairing (4.1) is right non-degenerate. That is, if $y \in H^1_f(F, T^*[\eta])$ is such that $\{x, y\}_\eta = 0$ for all $x \in \Phi$, then $y = 0$. In particular, if also*

- $\delta(T) = \mathfrak{d}(T) = A$;
- $(\eta T^*)^{G_F} = 0$;
- $(T^*/\eta T^*)^{G_F} = 0$;

*then the Bockstein pairing induces an injection*

$$H^1_f(F, T^*) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}_l}(\Phi, \mathbf{Q}_l/\mathbf{Z}_l).$$

Proof. Let $y$ be any element of $H^1_f(F, T^*[\eta])$ and let $c \in \Phi$ be the image of a class $\tilde{c} \in C^v \subseteq H^1(F, T)$ for some $v \in \mathcal{L}$. We compute the Bockstein pairing for these elements. To compute the pairing $\{c, y\}_\eta$ we first must lift $c$ via the map

$$H^1(F, T/\eta^2 T) \to H^1(F, T/\eta T)$$

to an element of $H^1(F, T/\eta^2 T)$ with singular restriction 0 away from some finite subset of places $\Sigma$. But this is easy; we simply take $\Sigma = \{v\}$ and lift $c$ to the image of $\tilde{c}$ in $H^1(F, T/\eta^2 T)$, which we also denote by $\tilde{c}$. The next step is to pull back $\tilde{c}_{v,s} \in H^1_s(F_v, T/\eta^2 T)$ under the injection

$$\eta : H^1_s(F_v, T/\eta T) \to H^1_s(F_v, T/\eta^2 T). \tag{4.2}$$

We denote this pull back by $\frac{1}{\eta}\tilde{c}_{v,s}$. The pairing $\{c, y\}_\eta$ is now nothing other than the Tate pairing of $\frac{1}{\eta}\tilde{c}_{v,s} \in H^1_s(F_v, T/\eta T)$ and $y_v \in H^1_f(F_v, T^*[\eta])$:

$$\{c, y\}_\eta = \left\langle \frac{1}{\eta}\tilde{c}_{v,s}, y_v \right\rangle_v . \tag{4.3}$$

Since $C^v$ has strict depth $\eta$ we can find classes $c \in \Phi$ such that the associated classes $\tilde{c}_{v,s}$ generate $\eta H^1_s(F_v, T/\eta^2 T)$ as an $A$-module. But this is just the image of $H^1_s(F_v, T/\eta T)$ under (4.2), so we see that the classes $\frac{1}{\eta}\tilde{c}_{v,s}$ generate $H^1_s(F_v, T/\eta T)$ as an $A$-module.

Now assume that $y \in H^1_f(F, T^*)$ is orthogonal to all of $\Phi$. (4.3) shows that $y_v$ is orthogonal to all of $H^1_s(F_v, T/\eta T)$ under the Tate pairing for all $v \in \mathcal{L}$. Since the Tate pairing is perfect, this implies that $y_v = 0$ for all $v \in \mathcal{L}$. Thus by Lemma 2.2 or Lemma 2.3 we have $y = 0$. This proves the non-degeneracy.

For the last injection, the non-degeneracy shows that $H^1_f(F, T^*[\eta])$ injects into $\mathrm{Hom}_{\mathbf{Z}_l}(\Phi, \mathbf{Q}_l/\mathbf{Z}_l)$. Lemma II.3.2 shows that $H^1_f(F, T^*[\eta]) = H^1_f(F, T^*)[\eta]$, and now Corollary 3.2 completes the proof. □

## 5. A $\delta$-vanishing result

In this section we give a proof of the following basic result on cohomology of the general linear group.

Proposition 5.1. *Let $A$ be an artin local ring with finite residue field $k$ of characteristic $l \neq 2$ and let $H$ be a free $A$-module of finite rank $n$. If $n = 2$ (resp. $n > 2$) then assume that $\#k \neq 5$ (resp. $l$ is at least 5 and does not divide $n + 1$). Then*

$$H^1\big(\mathrm{GL}_n(A), \mathrm{End}^0_A H\big) = 0.$$

Proof. Let $V$ be a free $k$-module of rank $n$. We first prove that

$$H^1\big(\mathrm{GL}_n(A), \mathrm{End}^0_k V\big) = 0 \tag{5.1}$$

by induction on the length of $A$; here $V$ is considered as a $\mathrm{GL}_N(A)$-module via the reduction map $\mathrm{GL}_N(A) \to \mathrm{GL}_N(k)$.

$A$ has length 1 precisely when $A = k$, and in this case [**DDT97**, Lemma 2.48] (for $n = 2$) and [**CPS75**, Table 4.5] (for $n > 2$) show that

$$H^1\big(\mathrm{SL}_n(k), \mathrm{End}^0_k V\big) = 0.$$

Since the index of $\mathrm{SL}_n(k)$ in $\mathrm{GL}_n(k)$ is prime to $l$, (5.1) follows immediately from this.

In the general case of (5.1), let $m$ be the largest integer such that $\mathfrak{m}^m \neq 0$, and consider the surjection

$$\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{m}^m).$$

Let $U$ denote the kernel, so that we have a short exact sequence

$$0 \to U \to \mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{m}^m) \to 0 \tag{5.2}$$

Note that $U = 1 + \mathfrak{m}^m M_n(k)$ as a subgroup of $\mathrm{GL}_n(A)$. In particular, $U$ is a $\mathrm{GL}_n(k)$-module and has the following decomposition into irreducible constituents:

$$U \cong (\mathrm{End}^0_k V)^r \oplus k^r; \tag{5.3}$$

here $r$ is the dimension of $\mathfrak{m}^m$ as a $k$-vector space.

Associated to (5.2) we have an inflation-restriction sequence

$$0 \longrightarrow H^1(\mathrm{GL}_n(A), \mathrm{End}_k^0 V) \longrightarrow H^1(\mathrm{GL}_n(A/\mathfrak{m}^m), \mathrm{End}_k^0 V) \longrightarrow$$

$$\mathrm{Hom}_{\mathrm{GL}_n(A/\mathfrak{m}^m)}(U, \mathrm{End}_k^0 V) \xrightarrow{\delta} H^2(\mathrm{GL}_n(A/\mathfrak{m}^m), \mathrm{End}_k^0 V)$$

Since $A/\mathfrak{m}^m$ has smaller length than $A$, by induction to prove (5.1) it suffices to show that $\delta$ is injective. Note that

$$\mathrm{Hom}_{\mathrm{GL}_n(A/\mathfrak{m}^m)}(U, \mathrm{End}_k^0 V) \cong \mathrm{Hom}_{\mathrm{GL}_n(k)}(U, \mathrm{End}_k^0 V)$$

since the $\mathrm{GL}_n(A/\mathfrak{m}^m)$-actions on both $U$ and $\mathrm{End}_k^0 V$ factor through $\mathrm{GL}_n(k)$.

Consider the exact sequence (5.2) as an element

$$c \in H^2(\mathrm{GL}_n(A/\mathfrak{m}^m), U).$$

Given an $f \in \mathrm{Hom}_{\mathrm{GL}_n(k)}(U, \mathrm{End}_k^0 V)$, by [**HS53**, Theorem 4] the image of $f$ under $\delta$ is nothing other than $f_* c$; that is, $\delta(f)$ is the the pushforward of (5.2) by $f$. To show that $\delta$ is injective, we must show that this pushforward splits only if $f$ is trivial.

One checks easily that none of the subextensions of $f_* c$ corresponding to copies of $\mathrm{End}_k^0 V$ in (5.3) split (for example, it suffices to exhibit two commuting matrices in $\mathrm{GL}_n(A/\mathfrak{m}^m)$ which lift to non-commuting matrices in $\mathrm{GL}_n(A)$). If $f \neq 0$, then by Schur's lemma there must be at least one factor of $\mathrm{End}_k^0 V$ in $U$ which maps isomorphically to $\mathrm{End}_k^0 V$ under $f$. But then the subextension of $f_* c$ corresponding to this factor of $\mathrm{End}_k^0 V$ does not split either, so $\delta(f) \neq 0$. This shows that $\delta$ is injective, and thus completes the proof of (5.1).

Lifting the result from $V$ to $H$ is easy: simply use the long exact sequence in $\mathrm{GL}_n(A)$-cohomology associated to the short exact sequence

$$0 \to \mathrm{End}_A^0(\mathfrak{m}H/\mathfrak{m}^t) \to \mathrm{End}_A^0(H/\mathfrak{m}^t) \to \mathrm{End}_A^0(V) \to 0$$

and an induction on $t$. □

As an immediate corollary we have the following $\delta$-vanishing result. We return now to the case of a finite, flat, local $\mathbf{Z}_l$-algebra $A$.

COROLLARY 5.2. *Let $H$ be an $l$-adic $G_F$-module over $A$ which is free of rank $n$ as an $A$-module and let $T = \mathrm{End}_A^0 H$. Assume that $l \neq 2$ and that the Galois representation $G_F \to \mathrm{Aut}_A H$ is surjective. If $n = 2$ (resp. $n > 2$), then assume that $\#k \neq 5$ (resp. $l \geq 5$ does not divide $n+1$). Then $\delta(T) = 0$.*

PROOF. Note that the surjectivity of $G_F \to \mathrm{Aut}_A H$ implies the surjectivity of $G_F \to \mathrm{Aut}_A H^*[\mathfrak{a}]$ for all ideals $\mathfrak{a}$ of $A$. The corollary thus follows immediately from Proposition 5.1. □

CHAPTER 4

# Flach systems

We begin this chapter by setting up the deformation theory of certain rank 2 Galois representations and relating it to Selmer groups. We then turn to the definitions and study of various notions of geometric Euler systems.

## 1. Minimally ramified deformations

We now turn to applications of the theory of the previous chapters to the deformation theory of Galois representations. We will consider deformation problems very similar to those considered in [**Wil95**, Chapter 1]. Let $l$ be an odd prime and let $A$ be a reduced, finite, flat, local $\mathbf{Z}_l$-algebra with maximal ideal $\mathfrak{m}$ and residue field $k$. Let $W(k)$ denote the Witt vectors of $k$; $A$ is canonically a $W(k)$-algebra. We now require $F$ to be a number field with at least one real embedding. We further require that $F_v$ is absolutely unramified for every $v \in \Sigma_l$.

Let $H$ be a free $A$-module of rank 2 with a continuous $A$-linear action of $G_F$. Fix an integer $k > l$. We make the following assumptions on this Galois representation:

- $H \otimes_A k$ is absolutely irreducible;
- $H$ is unramified away from a finite set of places $\Sigma$ containing $\Sigma_l$;
- For every $v \in \Sigma - \Sigma_l$, $H$ is minimally ramified at $v$ (see below); in particular, the inertia coinvariants $H_{\mathcal{I}_v}$ are free over $A$ for every $v \notin \Sigma_l$;
- $H$ is crystalline of weight $k$ at every place $v$ of $\Sigma_l$ (see below);
- There is a free $W(k)$-module $\widetilde{W}$ of rank 1 (with a chosen generator $\xi$) with a continuous $W(k)$-linear action of $G_F$ and a $A$-hermitian, Galois equivariant, perfect pairing
$$\psi : \tilde{A} \otimes_A H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l;$$
here $\tilde{A} = \widetilde{W} \otimes_{W(k)} A$;
- Every complex conjugation element in $G_F$ acts on $\widetilde{W}$ as multiplication by $-1$.
- The $W(k)$-algebra $A$ is generated by the traces of $\mathrm{Fr}(v)$ acting on $H_{\mathcal{I}_v}$ for all $v \notin \Sigma_l$.

We will say that such a Galois representation is of *Taylor-Wiles type of weight $k$*. Let $\chi : G_F \to W(k)^\times$ denote the inverse of the character of $\widetilde{W}$; $H$ has determinant $\chi$ over $A$. Set $\tilde{H} = \tilde{A} \otimes_A H$. By Lemma 4.1 the existence of $\psi$ implies that $A$ is a Gorenstein $\mathbf{Z}_l$-algebra. Fix a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$.

For each $v \notin \Sigma_l$ we define the *Hecke operator* $T_v \in A$ to be the trace of $\mathrm{Fr}(v)$ acting on $H_{\mathcal{I}_v}$. For $v \notin \Sigma$, we will write $\chi(v)$ for $\chi(\mathrm{Fr}(v))$; $\mathrm{Fr}(v)$ has characteristic polynomial
$$x^2 - T_v x + \chi(v)$$
for its action on $H$.

We say that $H$ is *minimally ramified* at a place $v$ if the image of the inertia group $\mathcal{I}_v$ in $\mathrm{GL}_2(A)$ is conjugate to one of the two subgroups

$$\left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right); b \in A\right\}, \qquad \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right); a \in A^\times\right\}.$$

Note in particular that in either case the inertia coinvariants $H_{\mathcal{I}_v}$ of $H$ are free over $A$ of rank 1, as asserted above. See [**Maz97**, Section 29] for more details.

For our crystalline conditions we must consider the integral theory of [**FL82**]; see also [**BK90**, Section 4]. Set

$$D_{\mathrm{cris}}(H) = H^0\big(F_v, B_{\mathrm{cris}} \otimes_{\mathbf{Z}_l} H\big).$$

We say that $H$ is *crystalline* at a place $v \in \Sigma_l$ if $H$ arises from a strongly divisible lattice $D$ in $D_{\mathrm{cris}}(H)$ via the Tate module functor; see [**BK90**, Theorem 4.3]. This implies in particular that

$$\dim_{F_v} D_{\mathrm{cris}}(H) = \mathrm{rank}_{\mathbf{Z}_l} H$$

so that $H \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is crystalline in the usual sense. $D$ is endowed with a decreasing filtration $F^i D$ by direct summands and we say that $H$ is of *weight* $k$ if

$$\mathrm{rank}_{\mathcal{O}_{F_v}} F^i D = \begin{cases} 2 & i \leq 0; \\ 1 & 1 \leq i \leq k-1; \\ 0 & k \leq i. \end{cases}$$

For an example of a representation of Taylor-Wiles type of weight 2, let $E$ be an elliptic curve over $F$ with good reduction away from $\Sigma - \Sigma_l$ and multiplicative reduction at every place of $\Sigma - \Sigma_l$. Then the $l$-adic Tate module $T_l E$ is a representation as above with $A = \mathbf{Z}_l$ and $\chi$ cyclotomic; see [**DDT97**, Section 2.2] for the case $F = \mathbf{Q}$. More generally, representations coming from modular forms or abelian varieties with real multiplication are often of this form. We will consider these examples in more detail later.

We are interested in deformations of $H \otimes_A k$ which satisfy the same local conditions as $H$. Specifically, let $\mathcal{C}$ denote the category of local noetherian $W(k)$-algebras with residue field $k$; a map between two rings in $\mathcal{C}$ is assumed to be local and to induce the identity map on $k$.

Following Diamond, if $B$ is an object of $\mathcal{C}$ and $H'$ is a free $B$-module of rank 2 with a $B$-linear action of $G_F$, we say that $H'$ is *minimally ramified* if

- $H'$ is unramified away from $\Sigma$;
- For every $v \in \Sigma - \Sigma_l$, $H' \otimes_A k$ is minimally ramified at $v$;
- $H'$ is crystalline at every place in $\Sigma_l$ in the sense of Fontaine-Laffaille and the filtration on the associated Dieudonné module $D$ satisfies $F^0 D = D$ and $F^k D = 0$;
- $H'$ has determinant $\chi$.

It is for the crystalline condition above for which we must assume that $k < l$m as otherwise the integral theory is not known. Let **Sets** denote the category of sets.

Definition 1.1. For any ring $B$ in $\mathcal{C}$, by a *minimally ramified lifting* of $H \otimes_A k$ we will mean a minimally ramified free $B$-module $H'$ of rank 2 with a continuous action of $G_F$ together with an isomorphism $\alpha' : H' \otimes_B k \cong H \otimes_A k$. We consider two such pairs $(H', \alpha')$, $(H'', \alpha'')$ to be isomorphic if there is an isomorphism $\beta : H' \cong H''$ as $B[G_F]$-modules such that $\alpha'' \beta \alpha'^{-1}$ is the identity map. We define a *minimally ramified deformation* of $H \otimes_A k$ to $B$ to be an isomorphism class of minimally ramified liftings of $H \otimes_A k$ to $B$. We define a covariant functor

$$D : \mathcal{C} \to \mathbf{Sets}$$

by letting $D(B)$ be the set of minimally ramified deformations of $H \otimes_A k$ to $B$. If $f : B \to B'$ is a morphism in $\mathcal{C}$, we let $f_* : D(B) \to D(B')$ be the map induced by $\cdot \otimes_B B'$ corresponding to $f$. We call $D$ the *minimally ramified deformation functor* of $H \otimes_A k$.

Results of Mazur and Ramakrishna show that the functor $D$ is representable by a $W(k)$-algebra $R$; that is, for any $B \in \mathcal{C}$, there is a functorial isomorphism

$$D(B) \cong \mathrm{Hom}_{W(k)-\mathrm{alg}}(R, B).$$

This isomorphism is therefore realized by a *universal minimally ramified deformation* $(H_R, \alpha_R) \in D(R)$ such that given a $W(k)$-algebra map $f : R \to B$, the pair $(H_R \otimes_R B, \alpha_R \otimes f)$ (with $B$ being regarded as an $R$-algebra via $f$) represents the corresponding isomorphism class in $D(B)$. See [**Maz90**, Section 1] for the basic representability result and [**Maz97**, Chapters 5 and 6] for a discussion of the extra deformation conditions. (Standard properties of crystalline representations show that the conditions at $\Sigma_l$ are categorical conditions in the sense of [**Maz97**, Section 25]; see [**Fon82**, Théorème of Section 5.2] and [**Fon94**, Section 0.1].)

We should comment that deformation functors are perhaps most naturally studied as a functor from the category of inverse limits of artinian local rings with residue fields $k$. (The inverse limits are computed in the category of topological rings.) See [**dSL97**] and [**Dic00**] for expositions. This is the category in which the universal deformation ring $R$ initially lives. In our cases $R$ will always be noetherian, so we will not concern ourselves with this distinction.

Note that $(H, \mathrm{id})$ is an element of $D(A)$; we therefore have a canonical map $\pi : R \to A$ such that $H \cong H_R \otimes_R A$. We will always regard $A$ as an $R$-algebra via $\pi$. For every place $v \notin \Sigma_l$, $\mathrm{Fr}(v)$ is defined acting on the inertia coinvariants $H_{R,\mathcal{I}_v}$ up to conjugation, and we define Hecke operators $\hat{T}_v \in R$ as the trace of $\mathrm{Fr}(v)$ acting on $H_{R,\mathcal{I}_v}$; these are all well-defined as before due to the minimal ramification hypotheses. Note that $\pi(\hat{T}_v) = T_v$; since we assumed that the $T_v$ generate $A$ as a $W(k)$-algebra, this implies that $\pi$ is surjective.

In order to study the ring $R$, we will primarily consider a variant of the functor $D$ which takes into account our initial representation $H$. Let $\mathcal{C}(A)$ be the category of local noetherian $W(k)$-algebras with residue field $k$ equipped with a local homomorphism $f$ to $A$ inducing the identity map on $k$:

$$\begin{array}{ccc} B & \twoheadrightarrow & k \\ {\scriptstyle f}\downarrow & & \| \\ A & \twoheadrightarrow & k \end{array}$$

A morphism in $\mathcal{C}(A)$ must respect these maps.

We define the *modified deformation functor*

$$D_A : \mathcal{C}(A) \to \mathbf{Sets}$$

as follows: Given an element $f : B \to A$ of $\mathcal{C}(A)$, we let $D_A(f : B \to A)$ be the inverse image of $(H, \mathrm{id}) \in D(A)$ under the map $f_* : D(B) \to D(A)$ induced by $f$.

That is, $D_A(f : B \to A)$ consists of the deformations of $H \otimes_A k$ to $B$ which are "congruent to $H$" via the augmentation homomorphism to $A$.

Since we assumed that $A$ is generated by the Hecke operators $T_v$, [**Maz97**, Section 20, Proposition 4] shows that $D_A$ is represented by $\pi : R \to A$ for the same ring and universal deformation $(H_R, \alpha_R)$ as $D$. Given our definition of morphisms in $\mathcal{C}(A)$, this means that for any object $f : B \to A$ in $\mathcal{C}(A)$, $D_A(f : B \to A)$ consists of those elements of $\mathrm{Hom}_{W(k)-\mathrm{alg}}(R, B)$ which yield $\pi$ on composition with $f$.

## 2. Tangent spaces and Selmer groups

The *Zariski tangent $A$-module* $t_{D_A}$ to the deformation functor $D_A$ is defined to be

$$t_{D_A} = D_A(A[\epsilon]).$$

Here $A[\epsilon] = A[\epsilon]/(\epsilon^2)$ and we take the map $A[\epsilon] \to A$ to be the natural map given by $\epsilon \mapsto 0$; in fact this is the only such map since $A$ is reduced. Since $R$ represents $D_A$, this means that $t_{D_A}$ consists of those elements of $\mathrm{Hom}_{W(k)-\mathrm{alg}}(R, A[\epsilon])$ which map to our distinguished homomorphism $\pi : R \to A$ on composition with the map $A[\epsilon] \to A$. $t_{D_A}$ does not encode information about torsion, however, so we shall work with certain other tangent spaces which carry more information. For any ideal $\mathfrak{a}$ of $A$, let $A_{\mathfrak{a}}$ be the ring $A[\epsilon]/(\mathfrak{a}\epsilon, \epsilon^2) = A \oplus \epsilon A/\mathfrak{a}$; we consider $A_{\mathfrak{a}}$ as an object of $\mathcal{C}(A)$ by using the $\epsilon \mapsto 0$ map as the augmentation to $A$. The following well known result connects the set $D_A(A_{\mathfrak{a}})$ to the module of continuous differentials $\Omega_R = \Omega_{R/W(k)}$ of $R$.

PROPOSITION 2.1. *For any ideal $\mathfrak{a}$ of $A$, there is a canonical isomorphism of $A$-modules*

$$D_A(A_{\mathfrak{a}}) \cong \mathrm{Hom}_A(\Omega_R \otimes_R A, A/\mathfrak{a}).$$

PROOF. The universal property of tensor products implies that there is a natural identification of $\mathrm{Hom}_A(\Omega_R \otimes_R A, A/\mathfrak{a})$ with $\mathrm{Hom}_R(\Omega_R, A/\mathfrak{a})$. This in turn identifies with the set $\mathrm{Der}_{W(k)}(R, A/\mathfrak{a})$ of $W(k)$-linear derivations from $R$ to $A/\mathfrak{a}$.

Given such a derivation $\omega : R \to A/\mathfrak{a}$, we obtain a homomorphism $f_\omega : R \to A_{\mathfrak{a}}$ by $f_\omega(r) = \pi(r) + \epsilon\omega(\partial r)$; $f_\omega$ obviously yields $\pi$ on mapping down to $A$. We therefore have defined a map $\mathrm{Der}_{W(k)}(R, A/\mathfrak{a}) \to D_A(A_{\mathfrak{a}})$ which is easily checked to be a map of $A$-modules. This map has an obvious inverse, sending an appropriate homomorphism $f : R \to A_{\mathfrak{a}}$ to the derivation given by $\omega_f(\partial r) = \frac{1}{\epsilon}(f(r) - \pi(r))$; this proves that it is an isomorphism. □

It is a fundamental result of deformation theory that the tangent module also has an interpretation as a certain Selmer group. We first need to introduce a Galois representation associated to $H$. Set

$$T = \mathrm{End}^0_A H(1).$$

$T$ is a free $A$-module of rank 3 with a continuous $A$-linear action of $G_F$. The isomorphism of Corollary B.5.3 yields an isomorphism of $A[G_F]$-modules

$$T \cong \tilde{A}(1) \otimes_A \mathrm{Sym}^2_A H.$$

LEMMA 2.2. *The canonical pairing*

$$\mathrm{End}^0_A H(1) \otimes \mathrm{End}^0_A H \to A(1) \to \mathbf{Z}_l(1)$$

*sending $f \otimes g$ to* tr *of the usual trace of $fg$ is a perfect pairing. In particular, it identifies $T^*$ with $\mathrm{End}^0_A H \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l$.*

Proof. That the pairing

$$\operatorname{End}^0_A H \otimes \operatorname{End}^0_A H \to A$$

sending $f \otimes g$ to the trace of $fg$ is perfect is a standard fact and is easily checked. The fact that composition with tr yields a perfect pairing to $\mathbf{Z}_l$ follows easily from Lemma B.3.1; twisting now gives the result. ☐

Let $\mathcal{S}$ be the finite/singular structure on $T$ which is minimally ramified away from $\Sigma_l$ (in particular, unramified away from $\Sigma$) and crystalline at all places of $\Sigma_l$. Since $H$ is assumed to be crystalline at all places above $l$, it follows easily that $T$ is crystalline at all such places. $T$ is therefore deRham as well and the discussion in Section I.5 shows that the structure $\mathcal{S}^*$ on $T^*$ also has the minimally ramified structure away from $\Sigma$ and the crystalline structure at all places of $\Sigma_l$. For any ideal $\mathfrak{a}$ of $A$, we also let $\operatorname{End}^0_A(H/\mathfrak{a}H)$ have the induced structure coming from its natural injection into $T^*$.

With respect to these finite/singular structure, we have the following result. This is essentially a standard isomorphism of deformation theory, and was proved in many cases in [**Wil95**, Propositions 1.2 and 1.3].

Proposition 2.3. *For any ideal $\mathfrak{a}$ of $A$, there is an isomorphism*

$$H^1_f\big(F, \operatorname{End}^0_A(H/\mathfrak{a}H)\big) \cong D_A(A_{\mathfrak{a}}).$$

Proof. We recall briefly the definition of this isomorphism; see [**Maz97**, Section 21] for more details. We have a distinguished element $\rho_0$ of $D_A(A_{\mathfrak{a}})$ coming from the natural injection $A \hookrightarrow A_{\mathfrak{a}}$. Given any other deformation $\rho' : G_F \to \mathrm{GL}_2(A_{\mathfrak{a}})$, (we have implicitly chosen a basis here in order to go from the deformation to an actual homomorphism) define a cocycle $c_{\rho'} : G_F \to \operatorname{End}_A(H/\mathfrak{a}H)$ by

$$c_{\rho'}(\sigma) = \tfrac{1}{\epsilon}(\rho_0^{-1}(\sigma)\rho'(\sigma) - 1).$$

(The expression in parentheses is divisible by $\epsilon$ since $\rho'$ is congruent to $\rho$ modulo $\epsilon$.) One checks that $c_{\rho'}$ really is a cocycle; that conjugating $\rho'$ changes $c_{\rho'}$ by a coboundary; and that this map from $D_A(A_{\mathfrak{a}})$ to $H^1(F, \operatorname{End}_A(H/\mathfrak{a}H))$ is injective.

We must show that the image is precisely $H^1_f(F, \operatorname{End}^0_A(H/\mathfrak{a}H))$. The restriction to trace zero matrices corresponds to our fixed determinant condition; see [**Maz97**, Section 24]. For the conditions at $v \notin \Sigma$, we first note that a representation of $G_F$ is unramified away from $\Sigma$ if and only if it factors through the maximal quotient of $G_F$ unramified away from $\Sigma$. This is the sort of Galois group to which deformation theory is usually applied, and Lemma II.4.1 shows that this notion of unramified is compatible with our local conditions $H^1_f(F_v, \operatorname{End}^0_A(H/\mathfrak{a}H))$.

For $v \in \Sigma - \Sigma_l$, the compatibility of our deformation condition and our cohomological condition is contained in [**Maz97**, Proposition of Section 29]. This leaves the crystalline conditions at $v \in \Sigma_l$; these are dealt with via the interpretation of $H^1(F_v, \operatorname{End}^0_A(H/\mathfrak{a}H))$ as extensions [**Maz97**, Section 22] together with [**BK90**, Lemma 4.5], which shows that the elements of $H^1(F_v, \operatorname{End}^0_A(H/\mathfrak{a}H))$ which correspond to crystalline extensions are precisely those which lie in the subgroup $H^1_f(F_v, \operatorname{End}^0_A(H/\mathfrak{a}H))$. ☐

Combining Proposition 2.1 and Proposition 2.3, we obtain an isomorphism

$$H^1_f\big(F, \operatorname{End}^0_A(H/l^nH)\big) \cong \operatorname{Hom}_A\big(\Omega_R \otimes_R A, A \otimes_{\mathbf{Z}_l} \mathbf{Z}/l^n\mathbf{Z}\big) \tag{2.1}$$

for any positive integer $n$. One checks easily that the formation of this Selmer group commutes with direct limits, so we obtain an isomorphism

$$H^1_f(F,T^*) \cong \mathrm{Hom}_A\big(\Omega_R \otimes_R A, A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l\big).$$

Our choice of Gorenstein trace tr and Lemma B.3.2 now yield an isomorphism

$$H^1_f(F,T^*) \cong \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_R \otimes_R A, \mathbf{Q}_l/\mathbf{Z}_l) \tag{2.2}$$

which will be fundamental to what follows.

## 3. Good primes

Let $\tau \in G_F$ be a fixed choice of complex conjugation; such a $\tau$ exists since we assumed that $F$ has at least one real embedding. By our assumption on the character $\chi$, the determinant of $\tau$ acting on $H$ is $-1$. It follows that $\tau$ acts as a non-scalar involution on $H$, and one checks directly from this that $\tau$ also acts as a non-scalar involution on $H^*$, $T$ and $T^*$.

Let $F_\mathfrak{m}$ denote the splitting field of $H \otimes_A k$. We let $\mathcal{L}$ denote the set of places of $F$ with Frobenius on $H \otimes_A k$ conjugate to $\tau$. Fix a place $v \in \mathcal{L}$. Since $\mathrm{Fr}(v)$ is conjugate to $\tau$ in $F_\mathfrak{m}$, we have the relation

$$\mathrm{Fr}(v)^2 - 1 \equiv 0 \pmod{\mathfrak{m}}; \tag{3.1}$$

this is the minimal polynomial of $\mathrm{Fr}(v)$ acting on $H \otimes_A k$ since $\mathrm{Fr}(v)$ is a non-scalar. We also have the characteristic polynomial

$$\mathrm{Fr}(v)^2 - T_v\,\mathrm{Fr}(v) + \chi(v) = 0$$

for the action of $\mathrm{Fr}(v)$ on $H$. We conclude from (3.1) that if $v \in \mathcal{L}$, then

$$T_v \equiv 0 \pmod{\mathfrak{m}}; \qquad \chi(v) \equiv -1 \pmod{\mathfrak{m}}. \tag{3.2}$$

We also have a factorization

$$x^2 - T_v x + \chi(v) \equiv (x-1)(x+1) \pmod{\mathfrak{m}}$$

coming from (3.1). $A$ is complete for the $\mathfrak{m}$-adic topology, so Hensel's lemma shows that this lifts to a factorization

$$x^2 - T_v x + \chi(v) = (x-\alpha)(x-\beta)$$

in $A[x]$ with

$$\alpha \equiv -\beta \equiv 1 \pmod{\mathfrak{m}}; \qquad \alpha+\beta = T_v; \qquad \alpha\beta = \chi(v).$$

Lemma 3.1. *Let $v$ be a place in $\mathcal{L}$. There is a direct sum decomposition (depending on $v$)*

$$H = H_\alpha \oplus H_\beta$$

*where $H_\alpha$ and $H_\beta$ are free of rank 1 over $A$, and $\mathrm{Fr}(v)$ acts on $H_\alpha$ as the scalar $\alpha$ and on $H_\beta$ as the scalar $\beta$.*

Proof. Set

$$H_\alpha = (\mathrm{Fr}(v)-\beta)H;$$
$$H_\beta = (\mathrm{Fr}(v)-\alpha)H.$$

Since $(\mathrm{Fr}(v)-\alpha)(\mathrm{Fr}(v)-\beta) = 0$, we see that $\mathrm{Fr}(v)$ acts on $H_\alpha$ as multiplication by $\alpha$ and on $H_\beta$ as multiplication by $\beta$. Furthermore,

$$H_\alpha \cap H_\beta = 0, \tag{3.3}$$

since $\mathrm{Fr}(v)$ acts on any element of this intersection simultaneously as 1 and $-1$ modulo $\mathfrak{m}$, and $2 \notin \mathfrak{m}$.

Let $h$ be an arbitrary element of $H$. Setting

$$h_\alpha = (\mathrm{Fr}(v) - \beta)h;$$
$$h_\beta = -(\mathrm{Fr}(v) - \alpha)h$$

we have $h_\alpha + h_\beta = (\alpha - \beta)h$. Since $\alpha - \beta \equiv 2 \pmod{\mathfrak{m}}$, it is a unit in $A$; this shows that $H_\alpha + H_\beta = H$. Combined with (3.3) this proves that we have a direct sum decomposition $H = H_\alpha \oplus H_\beta$.

It remains to show that both $H_\alpha$ and $H_\beta$ are free over $A$ of rank 1. Suppose first that $H_\alpha \otimes_A k = 0$. Then $\mathrm{Fr}(v)$ acts on all of $H \otimes_A k$ as the scalar $-1$; thus $\mathrm{Fr}(v)$ has trace congruent to $-2$ modulo $\mathfrak{m}$, which contradicts (3.2). We conclude that $H_\alpha \otimes_A k \neq 0$. In the same way we see that $H_\beta \otimes_A k \neq 0$. Since $H \otimes_A k$ is a two-dimensional vector space over $k$, it follows that $H_\alpha \otimes_A k$ and $H_\beta \otimes_A k$ are both one dimensional over $k$. Nakayama's lemma now implies that $H_\alpha$ and $H_\beta$ are cyclic as $A$-modules.

Since $H_\alpha$ and $H_\beta$ are cyclic $A$-modules, we have

$$\dim_{\mathbf{Q}_l} H_\alpha \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \leq \dim_{\mathbf{Q}_l} A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$$
$$\dim_{\mathbf{Q}_l} H_\beta \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \leq \dim_{\mathbf{Q}_l} A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l.$$

However, we also have

$$\dim_{\mathbf{Q}_l} (H_\alpha \oplus H_\beta) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = 2 \dim_{\mathbf{Q}_l} A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l.$$

It follows that

$$\dim_{\mathbf{Q}_l} H_\alpha \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = \dim_{\mathbf{Q}_l} H_\beta \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = \dim_{\mathbf{Q}_l} A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l.$$

Since $A$ is torsion-free over $\mathbf{Z}_l$, this implies that $H_\alpha$ and $H_\beta$ are both free of rank 1 over $A$, as claimed. ☐

The preceding result gives a very explicit characterization of the Galois representation $T$ at $v \in \mathcal{L}$ and makes it possible to compute the singular quotient $H^1_s(F_v, T)$.

LEMMA 3.2. *For all $v \in \mathcal{L}$, $H^1_s(F_v, T)$ is a free $A$-module of rank 1. For any ideal $\mathfrak{a}$ of $A$, $H^1_s(F_v, T/\mathfrak{a}T)$ is a free $A/\mathfrak{a}$-module of rank 1. In particular, the natural map*

$$H^1_s(F_v, T) \to H^1_s(F_v, T/\mathfrak{a}T)$$

*is surjective.*

PROOF. By Lemma 3.1 we can fix an $A$-basis $x, y$ of $H$ such that $\mathrm{Fr}(v)(x) = \alpha x$ and $\mathrm{Fr}(v)(y) = \beta y$; we have $\alpha\beta = \chi(v)$ and $\alpha \equiv -\beta \equiv 1 \pmod{\mathfrak{m}}$. This last congruence implies that $\alpha^2$ and $\beta^2$ are not equal to $\chi(v)$, since $\chi(v) \equiv -1 \pmod{\mathfrak{m}}$ by (3.2).

By Lemma I.1.3, $H^1_s(F_v, T) \cong T(-1)^{G_{k_v}} \cong (\mathrm{End}^0_A H)^{G_{k_v}}$. $G_{k_v}$ acts on $\mathrm{End}^0_A H$ as conjugation by $\mathrm{Fr}(v)$; that is, as conjugation by the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. This sends a matrix $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ to

$$\frac{1}{\chi(v)} \begin{pmatrix} \alpha\beta a & \alpha^2 b \\ \beta^2 c & -\alpha\beta a \end{pmatrix} = \begin{pmatrix} a & \frac{\alpha^2}{\chi(v)} b \\ \frac{\beta^2}{\chi(v)} c & -a \end{pmatrix}.$$

Since $\alpha^2$ and $\beta^2$ do not equal $\chi(v)$, it follows that $(\mathrm{End}^0_A H)^{G_{k_v}}$ is free of rank 1 over $A$, generated by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

The result for $H^1_s(F_v, T/\mathfrak{a}T)$ is proven in the same way, and the surjectivity follows immediately. □

## 4. Flach systems

In this section we introduce the weakest of the structures on the cohomology of Galois representations of Taylor-Wiles type which we will consider. We make the additional assumptions:

- $T \otimes_A k$ is absolutely irreducible;
- $H^1(F(T^*[\mathfrak{a}])/F, T^*[\mathfrak{a}]) = 0$ for every ideal $\mathfrak{a}$ of finite index in $A$.

DEFINITION 4.1. Let $\eta$ be a non-zero divisor in $A$. A *Flach system* $\{c^v\}_{v\in\mathcal{L}}$ *of depth* $\eta$ for $T$ is a weak Euler system $\{C^v\}_{v\in\mathcal{L}}$ of strict depth $\eta$ such that each $C^v$ is a cyclic $A$-module, generated by $c^v \in H^1(F,T)$.

Let $\{c^v\}_{v\in\mathcal{L}}$ be a Flach system for $T$ and let $\Phi$ denote the Euler module of the associated partial Euler system $\{C^v\}_{v\in\mathcal{L}}$. Note that essentially by definition the set $\mathcal{L}$ contains the set $\mathcal{L}_\tau$ of places associated to $T^*$ in Chapter III, Section 2. Lemma 3.2 and Corollary III.5.2 insure that $T$ and $\{C^v\}_{v\in\mathcal{L}}$ satisfy the hypotheses of Corollary III.3.2 with $\delta = \mathfrak{d} = A$; thus

$$\eta H^1_f(F, T^*) = 0.$$

In addition, Lemma III.2.2 shows that $\text{Ш}^1(F, T^*[\eta]) = 0$. The assumption that $T \otimes_A k$ is irreducible as a $G_F$-module implies that $(\eta T^*)^{G_F} = (T^*/\eta T^*)^{G_F} = 0$, so that by Lemma II.3.2 we have

$$H^1_f(F, T^*) = H^1_f\big(F, T^*[\eta]\big). \tag{4.1}$$

Combining all of this, we see that the Bockstein pairing as in Section III.4 exists:

$$H^1_f(F, T^*) \otimes_{\mathbf{Z}_l} H^1_f(F, T/\eta T) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

Proposition III.4.1 shows that it induces an injection

$$H^1_f(F, T^*) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}_l}(\Phi, \mathbf{Q}_l/\mathbf{Z}_l).$$

Note that this pairing depends on the choice of the generator $\eta$.

The isomorphism (2.2) and Pontrjagin duality now yield a surjection

$$\Phi \twoheadrightarrow \Omega_R \otimes_R A. \tag{4.2}$$

In particular, $\Omega_R \otimes_R A$ is $\eta$-torsion since $\Phi$ is by definition. In the case that $\eta$ is a unit, this completely determines $R$.

LEMMA 4.2. *Let $\pi : R \to A$ be a surjection of finite $W(k)$-algebras with residue field $k$ and suppose that $\Omega_R \otimes_R A = 0$. Suppose also that $W(k)$ injects into $A$. Then $\pi$ is an isomorphism, and both of $R$ and $A$ are isomorphic to $W(k)$.*

PROOF. Since $\Omega_R \otimes_R A = 0$ and $R$ is local, Nakayama's lemma implies that $\Omega_R = 0$. Consider the map $W(k) \to R$. Reducing modulo $l$ yields a map $k \to R/lR$. $R/lR$ is automatically flat over $k$, and $\Omega_{R/lR/k} = \Omega_{R/W(k)} \otimes R/lR = 0$; thus $R/lR$ is an étale local $k$-algebra with residue field $k$. But the only such algebra is $k$ itself. We conclude that $k \to R/lR$ is an isomorphism, and then by Nakayama's lemma that $W(k) \to R$ is a surjection. Since the map $W(k) \to A$ is an injection, this implies that $W(k) = R = A$, as claimed. □

To apply Lemma 4.2 in our case we need to know that $R$ is a finite $W(k)$-algebra. A standard argument in deformation theory shows that $R$ is surjected onto by a power series ring in $\dim_{\mathbf{F}_l} D(k[\varepsilon])$ variables. But $D(k[\varepsilon])$ is isomorphic to $H^1_f(F, \operatorname{End}^0_k(H \otimes_A k))$; the assumption that $\eta$ is a unit and the discussion above implies that the latter group is trivial. Thus $R$ is surjected onto by $W(k)$, so it is definitely finite. We state our result in this case as a proposition.

PROPOSITION 4.3. *If $T$ admits a Flach system of depth $\eta \in A^\times$, then the natural maps*

$$W(k) \to R \to A$$

*are all isomorphisms.*

In order to obtain analogous results when $\eta$ is not a unit we will need to impose more structure on our Flach system.

## 5. Cohesive Flach systems

We continue with the assumptions of the previous section.

DEFINITION 5.1. A *cohesive Flach system of depth $\eta$* for $T$ is a collection of classes $c^v \in H^1(F, T)$ for all $v \notin \Sigma_l$ such that

- $\{c^v\}_{v \in \mathcal{L}}$ is a Flach system of depth $\eta$;
- $c^v$ vanishes in $H^1_s(F_w, T)$ for all $v$ and all $w \neq v$;
- $c^v$ vanishes in $H^1_s(F_w, T/\eta T)$ for all places $v$ and $w$;
- The map $\Theta : A \to H^1(F, T/\eta T)$ sending $T_v$ to $c^v$ is well-defined and is a (continuous) derivation.

The third condition and the fact that the $T_v$ generate $A$ as a $W(k)$-module imply that the image of $\Theta$ actually lands in $H^1_f(F, T/\eta T)$. Note also that $\Theta$ is automatically $W(k)$-linear since $W(k)$ is unramified over $\mathbf{Z}_l$.

$\Theta$ induces an $A$-linear map

$$h : \Omega_A \to H^1_f(F, T/\eta T)$$

with image $\operatorname{im}\Theta$. Thus we have a surjection

$$\Omega_A \twoheadrightarrow \operatorname{im}\Theta. \tag{5.1}$$

Of course, there is an injection

$$\Phi \hookrightarrow \operatorname{im}\Theta. \tag{5.2}$$

We also have the surjection (4.2) induced by the Bockstein pairing:

$$\Phi \twoheadrightarrow \Omega_R \otimes_R A. \tag{5.3}$$

Lastly, we have a surjection

$$\Omega_R \otimes_R A \twoheadrightarrow \Omega_A \tag{5.4}$$

coming from the surjection $\pi : R \twoheadrightarrow A$. The existence of these four maps and [**Mat86**, Theorem 2.4] imply that all of them are isomorphism. We define the *Flach automorphism*

$$\Xi : \Omega_A \to \Omega_A$$

to be the composition of (5.1), the inverse of (5.2), (5.3) and (5.4).

Returning to the isomorphism (5.4), we see that this means that the surjection $\pi : R \to A$ induces an isomorphism on differentials. Such a map is said to be an *evolution*. In [**EM97**], it is shown that for many classes of rings $A$ (for example, local complete intersections), $A$ admits no non-trivial evolutions. In this case, one can conclude that $\pi$ is an isomorphism, and therefore that the Galois representation $H$ is the universal minimally ramified deformation of $H \otimes_A k$.

## 6. Cohesive Flach systems of Eichler-Shimura type

In order to give an explicit description of the map $\Xi$ introduced in the previous section, we will need an assumption about the behavior of the Flach classes $c^v$ in $H^1_s(F_v, T)$. We assume from now on that $v$ does not lie in $\Sigma$.

Recall that by Lemma I.1.3 we have a canonical identification of $H^1_s(F_v, T)$ with $T(-1)^{G_{k_v}} \cong (\operatorname{End}^0_A H)^{G_{k_v}}$. Explicitly, this isomorphism is realized as follows:

$$\begin{aligned} H^1_s(F_v, T) &\cong \operatorname{Hom}_{G_{k_v}}\big(\operatorname{Gal}(F_{v,\mathrm{s}}/F_v^{\mathrm{ur}}), T\big) \\ &\cong \operatorname{Hom}_{G_{k_v}}\big(\operatorname{Gal}(F_v^{\mathrm{ur}}(\lambda_0^{1/l^\infty})/F_v^{\mathrm{ur}}), T\big) \end{aligned}$$

where $\lambda_0$ is a fixed uniformizer of $F_v$ and the second isomorphism comes from the fact that $\operatorname{Gal}(F_v^{\mathrm{ur}}(\lambda_0^{1/l^\infty})/F_v^{\mathrm{ur}})$ is the maximal pro-$l$ quotient of $\operatorname{Gal}(F_{v,\mathrm{s}}/F_v^{\mathrm{ur}})$. Fix a $l^\infty$-root $\lambda$ of $\lambda_0$, in the sense of fixing a compatible $l^n$-th root $\lambda_n$ for each $n$. We will write $F_v^{\mathrm{ur}}(\lambda)$ for $F_v^{\mathrm{ur}}(\lambda_0^{1/l^\infty})$. Fix also a generator $\zeta$ of $\mathbf{Z}_l(1)$, and let $\zeta_n$ be the corresponding choice of generator of $\mu_{l^n}$. These choices determine a generator $\tau$ of $\operatorname{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}})$ by requiring $\tau(\lambda) = \zeta\lambda$, where this equation has the obvious meaning in terms of inverse limits. With these choices, $\operatorname{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) \cong \mathbf{Z}_l(1)$ as $G_{k_v}$-modules, where $\tau$ corresponds to $\zeta$. (We have indefinitely suspended the use of $\tau$ for a fixed complex conjugation.)

Let $\operatorname{Fr}(v)$ denote the endomorphism of $H$ induced by Frobenius. We define the *verschiebung* $\operatorname{Ver}(v)$ to be the endomorphism given by $\chi(v)\operatorname{Fr}(v)^{-1}$.

DEFINITION 6.1. A cohesive Flach system $\{c^v\}_{v\notin\Sigma_l}$ is said to be of *Eichler-Shimura type* of weight $2w$ if for each $v \notin \Sigma$, the class $c^v_{v,s} \in H^1_s(F_v, T)$ is given by

$$\begin{aligned} c^v_{v,s} : \operatorname{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) &\to T \\ \tau^j &\mapsto wj\eta(\operatorname{Fr}(v) - \operatorname{Ver}(v)) \otimes \zeta \end{aligned}$$

To check that $c^v_{v,s}$ really is a cocycle it suffices by Lemma I.1.3 to check that $\operatorname{Fr}(v) - \operatorname{Ver}(v)$ is $G_{k_v}$-equivariant; this is clear since $G_{k_v}$ is abelian. Note that $\operatorname{Fr}(v) - \operatorname{Ver}(v)$ really does have trace zero. One checks easily that this definition is independent of the choice of $\zeta$ and $\lambda$.

In the case that a cohesive Flach system is of Eichler-Shimura type we can compute the Flach automorphism completely explicitly.

THEOREM 6.2. *Assume that $T$ admits a cohesive Flach system of Eichler-Shimura type of depth $\eta$ and weight $2w$. Then the Flach automorphism $\Xi : \Omega_A \to \Omega_A$ is multiplication by $2w$.*

We will prove this result in the next chapter. Theorem 6.2 can be regarded as a sort of reciprocity law for the Bockstein pairing

$$H^1_f\big(F, T/\eta T\big) \otimes_{\mathbf{Z}_l} H^1_f\big(F, T^*[\eta]\big) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

By (2.2) and (4.1) the second term identifies with $\operatorname{Hom}_{\mathbf{Z}_l}(\Omega_R \otimes_R A, \mathbf{Q}_l/\mathbf{Z}_l)$ and then via the evolution $\pi$ with $\operatorname{Hom}_{\mathbf{Z}_l}(\Omega_A, \mathbf{Q}_l/\mathbf{Z}_l)$. The first term admits a map from $\Omega_A$ coming from the cohesive Flach system. In this context, Theorem 6.2 is precisely the following characterization of the Bockstein pairing.

COROLLARY 6.3. *The pairing*

$$\Omega_A \otimes_{\mathbf{Z}_l} \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_A, \mathbf{Q}_l/\mathbf{Z}_l) \to \mathbf{Q}_l/\mathbf{Z}_l$$

*induced from the Bockstein pairing by the maps described above is* $2w$ *times the canonical duality pairing.*

CHAPTER 5

# Flach systems of Eichler-Shimura type

In this chapter we give the proof of Theorem IV.6.2.

## 1. The map on differentials

We begin by recalling the details of the construction of the map $\Xi$. Fix a power $l^n$ of $l$ such that $\eta$ divides $l^n$ in $A$; such a power exists since $\eta$ is a non zero-divisor by Corollary B.2.2. By the existence of the cohesive Flach system and the irreducibility of $T \otimes_A k$, we have that

$$H^1_f(F, T^*) = H^1_f\big(F, T^*[\eta]\big) = H^1_f\big(F, T^*[l^n]\big).$$

For any $\mathbf{Z}_l$-module $M$ we will write $M^\vee$ for its *Pontrjagin dual* $\mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Q}_l/\mathbf{Z}_l)$; when $M$ is $l^n$-torsion this can be identified with $\mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}/l^n\mathbf{Z})$. We recall the definition of the map $\Xi$ in seven steps. We can and will work at finite levels since everything here is $l^n$-torsion.

(1) Let

$$\xi_1 : \Omega_A \to H^1_f(F, T/\eta T)$$

be the $A$-linear map induced by $\Theta$ and the universal property of $\Omega_A$. We have $\xi_1(\partial T_v) = c^v$.

(2) We have a Bockstein pairing

$$\{\cdot,\cdot\}_\eta : H^1_f(F, T/\eta T) \otimes_{\mathbf{Z}_l} H^1_f\big(F, T^*[\eta]\big) \to \mathbf{Q}_l/\mathbf{Z}_l$$

as in Section II.4. For the Flach classes $c^v \in H^1_f(F, T/\eta T)$, we computed in (III.4.3) that this pairing is given by

$$\{c^v, \kappa\}_\eta = \left\langle \tfrac{1}{\eta} c^v_{v,s}, \kappa_v \right\rangle_v .$$

Here $\kappa \in H^1_f(F, T^*[\eta])$ and

$$\langle \cdot,\cdot \rangle_v : H^1_s(F_v, T/\eta T) \otimes_{\mathbf{Z}_l} H^1_f\big(F_v, T^*[\eta]\big) \to \mathbf{Q}_l/\mathbf{Z}_l$$

is the restriction of the usual Tate pairing, which we will consider in more detail in the next section. Note that in the Tate pairing we can compute on the level of $l^n$-torsion rather than $\eta$-torsion. The Bockstein pairing thus yields a map

$$\xi_2 : H^1_f(F, T/\eta T) \to H^1_f\big(F, T^*[l^n]\big)^\vee.$$

(3) The pairing of Lemma IV.2.2 yields an isomorphism

$$\xi_3 : \mathrm{End}^0_A(H/l^n H) \xrightarrow{\simeq} T^*[l^n].$$

(4) Let

$$\rho_R : G_F \to \mathrm{GL}_2(R)$$

be a representative of the universal minimally ramified deformation of $H \otimes_A k$. Recall that our hypotheses on $H$ imply that there is a unique map $\pi : R \to A$ such that $\pi \circ \rho_R$ yields the deformation $H$. $\pi$ satisfies $\pi(\hat{T}_v) = T_v$ for all $v$.

We now use the isomorphism of (IV.2.1):

$$\xi_4 : \mathrm{Hom}_A(\Omega_R \otimes_R A, A/l^n A) \to H^1_f\big(F, \mathrm{End}^0_A(H/l^n H)\big);$$

here as always $\Omega_R = \Omega_{R/W(k)}$. We recall (with some additional motivation) how $\xi_4$ is defined: Let an $A$-linear map $\omega : \Omega_R \otimes_R A \to A/l^n A$ be given. Let $I$ be the kernel of the diagonal map $\Delta : R \otimes_{W(k)} R \to R$. There is a well-known isomorphism $\Omega_R \cong I/I^2$, under which $\partial r$ corresponds to the residue class of $r \otimes 1 - 1 \otimes r$. $\omega$ thus defines an $A$-linear map

$$\nu_1 : I \to A/l^n A$$

by sending $r \otimes 1 - 1 \otimes r$ to $\omega(\partial r \otimes 1)$.

The exact sequence

$$1 \to I \to R \otimes_{W(k)} R \to R \to 1$$

splits as an exact sequence of $R$-algebras (where we let $R$ act on the right factor in $R \otimes_{W(k)} R$) via the map sending $r \in R$ to $1 \otimes r$. This yields an isomorphism

$$R \otimes_{W(k)} R \cong R \oplus I \tag{1.1}$$

of $R$-algebras, where $r \otimes s$ corresponds to $(rs, r \otimes s - 1 \otimes rs)$. Restricting (1.1) to the left factor of $R$ we obtain a map

$$\nu_2 : R \to R \oplus I$$

sending $r$ to $(r, r \otimes 1 - 1 \otimes r)$.

Now let $A'$ be the ring

$$A' = A \oplus \epsilon A/l^n A = A[\epsilon]/(l^n \epsilon, \epsilon^2).$$

We define a map

$$\pi' : R \to A'$$

by composing $\nu_2$ with $\pi \oplus \epsilon \nu_1$. Thus

$$\pi'(r) = \pi(r) + \epsilon \omega(\partial r \otimes 1).$$

This in turn induces a representation

$$\rho' : G_F \to \mathrm{GL}_2(A')$$

by $\rho' = \pi' \circ \rho_R$; here $\rho_R$ is a representative of the universal minimally ramified deformation as above. We now define a cocycle

$$\kappa'_\omega : G_F \to \mathrm{End}^0_A(H/l^n H)$$
$$\sigma \mapsto \tfrac{1}{\epsilon}(\rho_A^{-1}(\sigma)\rho'(\sigma) - 1)$$

where $\rho_A = \pi \circ \rho_R$. We take $\kappa'_\omega$ to be the image of $\omega$ under $\xi_4$. The discussion in Section IV.2 shows that this map is independent of the choice of $\rho_R$ and respects the finite/singular structure.

In terms of our computations above, we find that if we write

$$\rho_R(\sigma) = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

(so that $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are functions of $\sigma$) then we have

$$\rho'(\sigma) = \begin{pmatrix} \pi(\hat{a}) + \epsilon\omega(\partial\hat{a}) & \pi(\hat{b}) + \epsilon\omega(\partial\hat{b}) \\ \pi(\hat{c}) + \epsilon\omega(\partial\hat{c}) & \pi(\hat{d}) + \epsilon\omega(\partial\hat{d}) \end{pmatrix}.$$

From here one computes that

$$\kappa'_\omega(\sigma) = \tfrac{1}{\pi(\hat{a}\hat{d}-\hat{b}\hat{c})} \begin{pmatrix} \pi(\hat{d})\omega(\partial\hat{a}) - \pi(\hat{b})\omega(\partial\hat{c}) & \pi(\hat{d})\omega(\partial\hat{b}) - \pi(\hat{b})\omega(\partial\hat{d}) \\ \pi(\hat{a})\omega(\partial\hat{c}) - \pi(\hat{c})\omega(\partial\hat{a}) & \pi(\hat{a})\omega(\partial\hat{d}) - \pi(\hat{c})\omega(\partial\hat{b}) \end{pmatrix}.$$

(5) The Gorenstein trace tr induces an isomorphism

$$\xi_5 : \mathrm{Hom}_A(\Omega_R \otimes_R A, A/l^n A) \to \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_R \otimes_R A, \mathbf{Z}/l^n\mathbf{Z})$$

sending $\omega$ to $\mathrm{tr} \circ \omega$; see Lemma B.3.2.

(6) There is a double duality isomorphism

$$\xi_6 : \Omega_R \otimes_R A \cong \big((\Omega_R \otimes_R A)^\vee\big)^\vee$$

sending $z \in \Omega_R \otimes_R A$ to the evaluation at $z$ map on $(\Omega_R \otimes_R A)^\vee$.

(7) There is a natural map

$$\xi_7 : \Omega_R \otimes_R A \to \Omega_A$$

sending $dr \otimes a$ to $a\partial\pi(r)$.

The map

$$\Xi : \Omega_A \to \Omega_A$$

is defined to be the composition

$$\begin{array}{ccccc} \Omega_A & \xrightarrow{\xi_1} & H^1_f(F, T/\eta T) & \xrightarrow{\xi_2} \\ H^1_f(F, T^*[l^n])^\vee & \xrightarrow{H(\xi_3)^\vee} & H^1_f\big(F, \mathrm{End}^0_A(H/l^n H)\big)^\vee & \xrightarrow{\xi_4^\vee} \\ \mathrm{Hom}_A(\Omega_R \otimes_R A, A/l^n A)^\vee & \xrightarrow{(\xi_5^{-1})^\vee} & \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_R \otimes_R A, \mathbf{Z}/l^n\mathbf{Z})^\vee & \xrightarrow{\xi_6^{-1}} \\ \Omega_R \otimes_R A & \xrightarrow{\xi_7} & \Omega_A & \end{array}$$

Note that the cohesive Flach system enters only into the very first map $\xi_1$; the remaining maps are at most dependent on the choice of Gorenstein trace tr, although one checks easily that the composite does not depend on that choice.

## 2. The Tate pairing

In order to explicitly compute the map $\Xi$ we will need to work with the Tate pairing. Let $M$ be a finite $G_{F_v}$-module of exponent $m$ and let $M^* = \mathrm{Hom}_{\mathbf{Z}}(M, \mu_m)$ be its Cartier dual. Recall that the Tate pairing is the map

$$H^1(F_v, M) \otimes H^1(F_v, M^*) \to \mathbf{Q}/\mathbf{Z}$$

defined as the composition of

$$\begin{array}{ccccc} H^1(F_v,M)\otimes H^1(F_v,M^*) & \xrightarrow{\text{cup}} & H^2(F_v,M\otimes_{\mathbf{Z}} M^*) & \xrightarrow{\text{Cartier}} & \\ H^2(F_v,\mu_m) & \xrightarrow{\simeq} & H^2(L/F_v,L^\times) & \xrightarrow{\text{val}} & \\ H^2(L/F_v,\mathbf{Z}) & \xleftarrow{\delta} & H^1(L/F_v,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\text{eval}} & \mathbf{Q}/\mathbf{Z}. \end{array}$$

Here $L$ is the unique unramified extension of $F_v$ of degree $m$. We recall in more detail the various maps involved. For yet more details, see [**Ser79**, Chapter 13, Section 3] or [**Mil86**, Chapter 1, Sections 1-2].

(1) The first map is simply cup product. One computes from the explicit formulas given in [**AW67**] that if $f : G_{F_v} \to M$ and $g : G_{F_v} \to M^*$ are cocycles, then a cocycle representing $f \cup g \in H^2(F_v, M \otimes_{\mathbf{Z}} M^*)$ is given by

$$(\sigma_1,\sigma_2) \mapsto f(\sigma_1)\otimes {}^{\sigma_1}g(\sigma_2). \tag{2.1}$$

(2) The next map is induced by Cartier duality between $M$ and $M^*$. Combined with the first map, the image of the pair $f, g$ is the cocycle

$$(\sigma_1,\sigma_2) \mapsto \sigma_1 g(\sigma_2)\big(\sigma_1^{-1} f(\sigma_1)\big) \in \mu_m$$

by definition of the adjoint action.

(3) The third map is the inverse of the isomorphism on $m$-torsion induced by the isomorphism

$$H^2(F_v^{\mathrm{ur}}/F_v, F_v^{\mathrm{ur}\,\times}) \to H^2(F_v, F_{v,\mathrm{s}}^\times) \leftarrow H^2(F_v,\mu_\infty)$$

where the first map is inflation and the second is induced by the inclusion $\mu_\infty \hookrightarrow F_{v,\mathrm{s}}^\times$. (See [**Ser79**, Chapter 13, Section 3].) This map seems to be quite difficult to describe explicitly; in our computation we will get lucky.

(4) The next map is simply the map induced by the valuation map on $L^\times$; this has the obvious interpretation on cocycles.

(5) The next map is the connecting homomorphism in the long exact sequence of $\mathrm{Gal}(L/F_v)$-cohomology coming from the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

It is an isomorphism since $\mathbf{Q}$ is divisible. One computes from the construction of the connecting homomorphism that if $f : \mathrm{Gal}(L/F_v) \to \mathbf{Q}/\mathbf{Z}$ is a 1-cocycle, then $\delta(f)$ is the 2-cocycle given by

$$\delta(f)(\sigma_1,\sigma_2) = \tilde f(\sigma_1\sigma_2) - \tilde f(\sigma_1) - \tilde f(\sigma_2) \in \mathbf{Z}$$

where $\tilde f : \mathrm{Gal}(L/F_v) \to \mathbf{Q}$ is any lifting of $f$. (Here we have used the fact that $\mathrm{Gal}(L/F_v)$ acts trivially on $\mathbf{Q}$.)

(6) The last map is evaluation at $\mathrm{Fr}(v) \in \mathrm{Gal}(L/F_v)$; this makes sense as $\mathbf{Q}/\mathbf{Z}$ has trivial action, so that $H^1(L/F_v,\mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(\mathrm{Gal}(L/F_v),\mathbf{Q}/\mathbf{Z})$.

In our computation we will follow the Tate pairing as far as $H^2(L/F_v, L^\times)$. Let us compute the images in this group of $\frac{e}{m} \in \mathbf{Q}/\mathbf{Z}$ in order to have something to compare with. $\frac{e}{m}$ corresponds under eval to the homomorphism $f_e : \mathrm{Gal}(L/F_v) \to \mathbf{Q}/\mathbf{Z}$ given by $f_e(\mathrm{Fr}(v)^i) = \frac{ei}{m}$. Let $\{\cdot\} : \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}$ be the map sending $x \in \mathbf{Q}/\mathbf{Z}$ to the unique $\tilde x \in \mathbf{Q}$ such that $0 \le \tilde x < 1$ and $x \equiv \tilde x \pmod{\mathbf{Z}}$. Let $\tilde f_e : \mathrm{Gal}(L/F_v) \to \mathbf{Q}$ be the lifting of $f_e$ given by $\tilde f_e(\mathrm{Fr}(v)^i) = \{\frac{ei}{m}\}$.

Under the map $\delta$ we now obtain the 2-cocycle

$$\delta(\tilde{f}_e)(\mathrm{Fr}(v)^i, \mathrm{Fr}(v)^j) = \left\{\frac{(i+j)e}{m}\right\} - \left\{\frac{ie}{m}\right\} - \left\{\frac{je}{m}\right\}$$

in $H^2(L/F_v, \mathbf{Z})$. Let $\lambda_0$ be our fixed uniformizer of $F_v$. Using that $\mathrm{Gal}(L/F_v)$ acts trivially on $\lambda_0$ and that $\lambda_0$ is a uniformizer in $L$, we see that the cocycle $C_e \in H^2(L/F_v, L^\times)$ given by

$$C_e(\mathrm{Fr}(v)^i, \mathrm{Fr}(v)^j) = \lambda_0^{\left\{\frac{(i+j)e}{m}\right\} - \left\{\frac{ie}{m}\right\} - \left\{\frac{je}{m}\right\}} \tag{2.2}$$

maps to $\delta(\tilde{f}_e)$ under $v$. Thus $C_e$ corresponds to $\frac{e}{m} \in \mathbf{Q}/\mathbf{Z}$ under $\mathrm{eval} \circ \delta^{-1}$; this is the cocycle we will use for comparison later.

In our computation, we will actually consider the induced pairing

$$H^1_s(F_v, M) \otimes H^1_f(F_v, M^*) \to \mu_m.$$

To compute with this pairing, one must first lift the cocycle in $H^1_s(F_v, M)$ to a cocycle in $H^1(F_v, M)$; the rest of the definition is the same. The fact that the other cocycle is finite implies that the pairing is independent of the choice of lifting.

## 3. A special case

**3.1. Additional hypotheses.** In this section we compute $\Xi(\partial T_v)$ with some additional simplifying hypotheses; this computation will still contain most of the content of the general case, but it is significantly simpler algebraically.

We make two assumptions. First, assume that the action of $\mathrm{Fr}(v)$ on $H$ is diagonal with respect to a fixed basis $x, y$; that is, $\mathrm{Fr}(v)$ acts on $H$ by a matrix

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

with $u, v \in A$. In particular, $uv = \chi(v)$ and $u + v = T_v$. (We apologize for the use of $v$ in two completely different ways; we hope that it will cause no confusion.) It follows from the definition of a cohesive Flach system of Eichler-Shimura type that in this case that the cocycle $c^v_{v,s}$ is given by

$$\begin{aligned} c^v_{v,s} : \mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) &\to T \\ \tau^j &\mapsto wj\eta(u-v) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \zeta_n \end{aligned} \tag{3.1}$$

with the notation of Chapter IV.7; $\zeta_n$ is the primitive $l^n$-root of unity induced by $\zeta$.

The second simplifying assumption is that the map $\pi : R \to A$ is an isomorphism; that is, $A$ is the universal minimally ramified deformation ring of $H \otimes_A k$ and $H$ is the universal deformation. Of course, we will identify $R$ with $A$ via $\pi$.

**3.2. Preliminaries.** To compute $\Xi(T_v)$, we begin by computing the image of $\partial T_v$ in $\mathrm{Hom}_A(\Omega_A, A/l^n A)^\vee$. (Recall that $R = A$, so that $\Omega_R \otimes_R A = \Omega_A$.) So let

$$\omega : \Omega_A \to A/l^n A$$

be a fixed map; we will compute its image in $\mathbf{Z}/l^n\mathbf{Z}$ under

$$\xi_4^\vee \circ H(\xi_3)^\vee \circ \xi_2 \circ \xi_1(\partial T_v) \in \mathrm{Hom}_A(\Omega_A, A/l^n A)^\vee. \tag{3.2}$$

**3.3.** $\xi_4$**.** Using the definition of $\xi_4$, the image of $\omega$ under (3.2) is the same as the image under

$$H(\xi_3)^\vee \circ \xi_2 \circ \xi_1(\partial T_v) \in H^1_f(F, \mathrm{End}^0_A(H/l^nH))^\vee \tag{3.3}$$

of the cohomology class represented by the cocycle

$$\kappa' : G_F \to \mathrm{End}^0_A(H/l^nH)$$

given by

$$\kappa'(\sigma) = \tfrac{1}{ad-bc}\begin{pmatrix} d\omega(\partial a) - b\omega(\partial c) & d\omega(\partial b) - b\omega(\partial d) \\ a\omega(\partial c) - c\omega(\partial a) & a\omega(\partial d) - c\omega(\partial b) \end{pmatrix}.$$

Here $\sigma \in G_F$ and

$$\rho_A(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A).$$

**3.4.** $\xi_3$**.** Using the definition of $\xi_3$, the image of $\kappa_\omega$ under (3.3) is the same as the image under

$$\xi_2 \circ \xi_1(\partial T_v) \in H^1_f(F, T^*[l^n])^\vee$$

of the cohomology class represented by the cocycle

$$\begin{aligned}\kappa : G_F \to T^*[l^n] &= \mathrm{Hom}_{\mathbf{Z}_l}\big(\mathrm{End}^0_A(H/l^nH)(1), \mu_{l^n}\big) \\ &= \mathrm{Hom}_{\mathbf{Z}_l}\big(\mathrm{End}^0_A(H/l^nH), \mathbf{Z}/l^n\mathbf{Z}\big)\end{aligned}$$

where $T^*[l^n]$ is identified with $\mathrm{End}^0_A(H/l^nH)$ via Lemma IV.2.2. Explicitly, we find that

$$\begin{aligned}\kappa(\sigma)&\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \\ &= \mathrm{tr}\left(\mathrm{trace}\left(\tfrac{1}{ad-bc}\begin{pmatrix} d\omega(\partial a) - b\omega(\partial c) & d\omega(\partial b) - b\omega(\partial d) \\ a\omega(\partial c) - c\omega(\partial a) & a\omega(\partial d) - c\omega(\partial b) \end{pmatrix} \cdot \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right)\right)\right) \\ &= \mathrm{tr}\Big(\tfrac{1}{ad-bc}\big(\alpha d\omega(\partial a) - \alpha b\omega(\partial c) + \gamma d\omega(\partial b) - \gamma b\omega(\partial d) + \\ &\qquad \beta a\omega(\partial c) - \beta c\omega(\partial a) - \alpha a\omega(\partial d) + \alpha c\omega(\partial b)\big)\Big).\end{aligned} \tag{3.4}$$

(Keep in mind that we have both a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ and the usual trace of linear algebra.)

**3.5.** $\xi_2$**.** Using the definition of $\xi_2$ and its explicit expression for $c^v = \xi_1(\partial T_v)$, we find that the desired element of $\mathbf{Z}/l^n\mathbf{Z}$ is the value of the Tate pairing

$$\left\langle \tfrac{1}{\eta} c^v_{v,s}, \kappa_v \right\rangle_v;$$

here we are identifying the image $\frac{1}{l^n}\mathbf{Z}/\mathbf{Z}$ with $\mathbf{Z}/l^n\mathbf{Z}$. It remains to compute this.

**3.6. The Tate pairing : preliminaries.** To begin with, note that $\kappa_v$ factors through $\mathrm{Gal}(F_v^{\mathrm{ur}}/F_v)$, as it is unramified at $v$. It follows that we are only interested in $\kappa(\mathrm{Fr}(v)^i)$. Using the fact that

$$\rho_A(\mathrm{Fr}(v)^i) = \begin{pmatrix} u^i & 0 \\ 0 & v^i \end{pmatrix}$$

which has determinant $\chi(v)^i$, we find that (3.4) simplifies to

$$\kappa(\mathrm{Fr}(v)^i)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(\chi(v)^{-i}\alpha\left(v^i\omega(\partial u^i) - u^i\omega(\partial v^i)\right)\right). \tag{3.5}$$

Next, note that

$$u^i\partial v^i = u^i v^{i-1} i\partial v = \chi(v)^{i-1} u i \partial v$$

and similarly

$$v^i\partial u^i = \chi(v)^{i-1} v i \partial u.$$

We therefore can write (3.5) as

$$\kappa(\mathrm{Fr}(v)^i)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(i\chi(v)^{-1}\alpha\left(v\omega(\partial u) - u\omega(\partial v)\right)\right).$$

Setting $K = F_v(H/l^nH)$ (so that $K/F_v$ is unramified), we see that $\kappa$ factors through $\mathrm{Gal}(K/F_v)$.

For $c^v$, we computed in (3.1) that

$$c^v_{v,s} : \mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) \to T/l^nT$$
$$\tau^j \mapsto wj\eta(u-v)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \zeta_n$$

Since $\tau^{l^n}$ goes to 0 under this map, $c^v$ factors through $\mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda_n)/F_v^{\mathrm{ur}})$; here by $\lambda_n$ we mean the $l^n$-th root of $\lambda_0$ determined by our earlier choice of $\lambda$.

In order to compute the Tate pairing of $\kappa$ and $c^v$ we first must descend $c^v$ to a cocycle over $F_v$. We can do this over the field $K(\lambda_n)$ as follows: let $G = \mathrm{Gal}(K(\lambda_n)/F_v)$. Denote by $\varphi$ the element of $G$ which acts as Frobenius on $K$ and fixes $\lambda_n$, and denote by $\tau$ the element of $G$ which is the identity on $K$ and sends $\lambda_n$ to $\zeta_n\lambda_n$. Then $\varphi$ and $\tau$ generate $G$ with the relations

$$\varphi^{[K:F_v]} = \tau^{l^n} = 1, \qquad \tau\varphi = \varphi\tau^{\varepsilon(v)}.$$

Note that $\tau^{\varepsilon(v)}$ makes sense as $\tau$ is $l^n$-torsion. We can represent $\frac{1}{\eta}c^v_{v,s}$ by the map

$$\theta^v : G \to T/l^nT$$
$$\varphi^i\tau^j \mapsto w\varepsilon(v)^i j(u-v)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \zeta_n.$$

One easily checks that this really is a cocycle and that $\eta\theta^v$ restricts to $c^v_{v,s}$ in $H^1_s(K(\lambda_n)/F_v, T/l^nT)$.

Via inflation we can represent $\kappa_v$ by the cocycle

$$\kappa_v : G \to T^*[l^n]$$

given by

$$\kappa_v(\varphi^i\tau^j)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(i\chi(v)^{-1}\alpha(v\omega(\partial u) - u\omega(\partial v))\right). \tag{3.6}$$

**3.7. The Tate pairing : cup product.** We can now compute the Tate pairing. The first step is to form the cup product

$$\theta^v \cup \kappa_v \in H^2\big(G, T/l^nT \otimes_{\mathbf{Z}_l} T^*[l^n]\big).$$

By (2.1), we see that $\theta^v \cup \kappa_v$ sends the pair $(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) \in G \times G$ to

$$\theta^v(\varphi^i\tau^j) \otimes \kappa_v^{\varphi^i\tau^j}(\varphi^{i'}\tau^{j'}) \in T/l^nT \otimes_{\mathbf{Z}_l} T^*[l^n].$$

Under Cartier duality this maps to the cocycle $C \in H^2(G, \mu_{l^n})$ given by

$$\begin{aligned} C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) &= \kappa_v^{\varphi^i\tau^j}(\varphi^{i'}\tau^{j'})\big(\theta^v(\varphi^i\tau^j)\big) \\ &= \kappa_v^{\varphi^i\tau^j}(\varphi^{i'}\tau^{j'})\left(w\varepsilon(v)^i j(u-v)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\zeta_n. \end{aligned}$$

Recall that $\kappa_v(\varphi^{i'}\tau^{j'})$ is a map

$$\operatorname{End}_A^0(H/l^nH) \to \mathbf{Z}/l^n\mathbf{Z}.$$

In particular, $\varphi^i\tau^j$ acts trivially on both the domain and the range. Thus by the definition of the adjoint Galois action we find that

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \operatorname{tr}\left(i'\chi(v)^{-1}w\varepsilon(v)^i j(u-v)(v\omega(\partial u) - u\omega(\partial v))\right)\zeta_n$$

If we let $C_0 : G \times G \to \mu_{l^n}$ be the cocycle

$$C_0(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \varepsilon(v)^i i' j \zeta_n,$$

then we conclude that $\omega$ maps to

$$\operatorname{tr}\left(\omega(w\chi(v)^{-1}(v\partial u - u\partial v))\right) I \tag{3.7}$$

where $I$ is the image of $C_0$ under the invariant map

$$H^2(K(\lambda_n)/F_v, \mu_{l^n}) \to \mathbf{Z}/l^n\mathbf{Z}.$$

**3.8. $\xi_5$ and $\xi_6$.** At this point, thankfully, we get the maps $\xi_5$ and $\xi_6$ for free. Specifically, suppose that we began with

$$\omega_0 : \Omega_A \to \mathbf{Z}/l^n\mathbf{Z}$$

and wished to compute its image in $\mathbf{Z}/l^n\mathbf{Z}$ under $(\xi_5^{-1})^\vee \cdots \xi_1(\partial T_v)$. By the definition of $\xi_5$ this would be the same as the image under $\xi_4^\vee \cdots \xi_1(\partial T_v)$ of the unique $\omega : \Omega_A \to A/l^nA$ such that $\operatorname{tr} \circ \omega = \omega_0$. But by (3.7) this is visibly just

$$\omega_0\big(w\chi(v)^{-1}(u-v)(v\partial u - u\partial v)\big)I. \tag{3.8}$$

Similarly, in $\operatorname{Hom}_{\mathbf{Z}_l}(\Omega_A, \mathbf{Z}/l^n\mathbf{Z})^\vee$ (3.8) is just the evaluation at

$$w\chi(v)^{-1}(u-v)(v\partial u - u\partial v)I \tag{3.9}$$

map, so that (3.9) is the final image of $\partial T_v$ in $\Omega_A$. It remains, then, to compute $I$ and to simplify our expression.

**3.9. Computation of $I$.** We begin by computing $I$; it is the image of $C_0$ under the maps

$$H^2(K(\lambda_n)/F_v, \mu_{l^n}) \to H^2(L/F_v, L^\times) \to \mathbf{Z}/l^n\mathbf{Z},$$

where $L$ is the unique unramified extension of $F_v$ of degree $l^n$. We first need to modify $C_0$ by a coboundary to get it to factor through $\mathrm{Gal}(L/F_v)$ and to take values in $L^\times$. We can do this using the cochain $f : G \to K(\lambda_n)^\times$ given by

$$f(\varphi^i\tau^j) = \lambda_n^{\langle i\rangle}$$

where $\langle i\rangle$ is the unique integer in $\{0, 1, \ldots, l^n - 1\}$ which is congruent to $i$ modulo $l^n$. The coboundary formula states that

$$C_0\partial f(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \frac{C_0(\varphi^i\tau^j, \varphi^{i'}\tau^{j'})f(\varphi^i\tau^j\varphi^{i'}\tau^{j'})}{{}^{\varphi^i\tau^j}f(\varphi^{i'}\tau^{j'})f(\varphi^i\tau^j)}$$

One computes easily that $\varphi^i\tau^j\varphi^{i'}\tau^{j'} = \varphi^{i+i'}\tau^{j''}$ for some $j''$, so that we can compute this as

$$\begin{aligned}
C_0\partial f(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) &= \frac{\zeta_n^{\varepsilon(v)^i i' j}\lambda_n^{\langle i+i'\rangle}}{\varphi^i\tau^j(\lambda_n^{\langle i'\rangle})\lambda_n^{\langle i\rangle}} \\
&= \frac{\zeta_n^{\varepsilon(v)^i i' j}\lambda_n^{\langle i+i'\rangle}}{\varphi^i(\zeta_n^{j\langle i'\rangle}\lambda_n^{\langle i'\rangle})\lambda_n^{\langle i\rangle}} \\
&= \frac{\zeta_n^{\varepsilon(v)^i i' j}\lambda_n^{\langle i+i'\rangle}}{\zeta_n^{\varepsilon(v)^i j\langle i'\rangle}\lambda_n^{\langle i'\rangle+\langle i\rangle}} \\
&= \lambda_n^{\langle i+i'\rangle-\langle i\rangle-\langle i'\rangle}.
\end{aligned}$$

This, however, is simply the inflation to $H^2(K(\lambda_n)/F_v, K(\lambda_n)^\times)$ of the cocycle $C_1 \in H^2(L/F_v, L^\times)$ of (2.2). Since $C_1$ was defined to map to 1 under the invariant map, we see that $C_0$ does as well. Thus $I = 1$.

**3.10. Differentials and Hecke operators.** We conclude by (3.9) that

$$\Xi(\partial T_v) = w\chi(v)^{-1}(u - v)(v\partial u - u\partial v) \in \Omega_A.$$

It remains to simplify this expression. Using that $uv = \chi(v)$ we find that

$$\begin{aligned}
(u - v)(v\partial u - u\partial v) &= \left(u - \tfrac{\chi(v)}{u}\right)\left(\tfrac{\chi(v)}{u}\partial u - u\partial\tfrac{\chi(v)}{u}\right) \\
&= \left(u - \tfrac{\chi(v)}{u}\right)\left(\chi(v)\tfrac{\partial u}{u} + \chi(v)\tfrac{\partial u}{u}\right) \\
&= \left(u - \tfrac{\chi(v)}{u}\right)\left(2\chi(v)\tfrac{\partial u}{u}\right) \\
&= 2\chi(v)\left(\partial u - \chi(v)\tfrac{\partial u}{u^2}\right) \\
&= 2\chi(v)\left(\partial u + \partial v\right).
\end{aligned}$$

Thus we conclude that

$$\Xi(\partial T_v) = 2w\partial(u + v) = 2w\partial T_v.$$

This completes the proof of Theorem IV.6.2 in this case.

## 4. A matrix computation

The key to removing both of the assumptions of the previous computation is the following matrix lemma.

LEMMA 4.1. *Let $R$ be a ring, $S$ an $R$-algebra and $M$ an $S$-module. Let $\partial : S \to M$ be an $R$-linear derivation. Let*

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(S)$$

*be a matrix with determinant $\delta = ad - bc$ and trace $t = a + d$. Assume that $\delta$ lies in the image of $R$ in $S$. Let $e$ be a positive integer, and write*

$$T^e = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(S).$$

*Then*

$$\begin{aligned} -2e\delta^e\partial t = (2bC - aD + dD)\partial A + (-2cD - aC + dC)\partial B + \\ (-2bA + aB - dB)\partial C + (2cB + aA - dA)\partial D \end{aligned} \tag{4.1}$$

PROOF. We prove (4.1) by induction on $e$, with the case $e = 0$ being trivial. Suppose then that we know (4.1) for $e$. We have

$$T^{e+1} = T^e T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aA + cB & bA + dB \\ aC + cD & bC + dD \end{pmatrix}. \tag{4.2}$$

Let $Z$ be the value of the expression on the right in the statement of the lemma for $e + 1$. After some simplification, one finds from (4.2) that

$$\begin{aligned} Z = \big(tbC + (-2\delta + td)D\big)\partial(aA + cB) + \big(-tcD + (2\delta - ta)C\big)\partial(bA + dB) + \\ \big(-tbA + (2\delta - td)B\big)\partial(aC + cD) + \big(tcB + (-2\delta + ta)A\big)\partial(bC + cD). \end{aligned}$$

Expanding this out with the product rule one obtains

$$\begin{aligned} Z = \big(\delta tD + 2\delta(bC - aD)\big)\partial A + \big(-\delta tC + 2\delta(dC - cD)\big)\partial B + \\ \big(-\delta tB + 2\delta(aB - bA)\big)\partial C + \big(\delta tA + 2\delta(cB - dA)\big)\partial D + \\ \delta'\big((td - 2\delta)\partial a - tc\partial b - tb\partial c + (ta - 2\delta)\partial d\big) \end{aligned} \tag{4.3}$$

where $\delta' = AD - BC = \delta^e$.

Now, since $ad - bc = \delta \in R$ and $\partial$ is $R$-linear, we have

$$a\partial d + d\partial a - b\partial c - c\partial b = 0.$$

Similarly,

$$A\partial D + D\partial A - B\partial C - C\partial B = 0. \tag{4.4}$$

Using these relations (4.3) simplifies to

$$\begin{aligned} Z = \delta\big((2bC - 2aD)\partial A + (2dC - 2cD)\partial B + \\ (2aB - 2bA)\partial C + (2cB - 2dA)\partial D\big) - 2\delta'\delta(\partial a + \partial d). \end{aligned} \tag{4.5}$$

Multiplying (4.4) by $t\delta$ yields

$$\delta\big((aD + dD)\partial A + (-aC - dC)\partial B + (-aB - dB)\partial C + (aA + dA)\partial D\big) = 0.$$

Adding this to (4.5), we find that

$$Z = \delta\big((2bC - aD + dD)\partial A + (-2cD - aC + dC)\partial B + (-2bA + aB - dB)\partial C + (2cB + aA - dA)\partial D\big) - 2\delta'\delta(\partial a + \partial d).$$

The induction hypothesis shows that this is just

$$Z = \delta(-2e\delta^e \partial t) - 2\delta\delta'(\partial t) = -2(e+1)\delta^{e+1}\partial t.$$

This completes the induction. □

## 5. Computation of Ξ in the non-diagonal case

We now explain how to compute $\Xi(\partial T_v)$ when $\mathrm{Fr}(v)$ is not necessarily diagonal. We continue to assume that $\pi$ is an isomorphism. This computation is fundamentally the same as the previous special case, just a bit messier and with the simple computation of Section 3.10 replaced by the more elaborate computation of Lemma 4.1.

The complication is that $\mathrm{Fr}(v)$ no longer acts diagonally. Write

$$\rho_A(\mathrm{Fr}(v)) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\rho_A(\mathrm{Fr}(v)^i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

for some fixed basis $x, y$ of $H$. Note that $a+d = T_v$, $ad-bc = \chi(v)$ and $a_i d_i - b_i c_i = \chi(v)^i$. The formula for the cocycle $\kappa$ is exactly as computed in (3.4), replacing $a, b, c, d$ with $a_i, b_i, c_i, d_i$ for $\sigma = \mathrm{Fr}(v)^i$. The Flach class is now

$$c^v_{v,s} : \mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) \to T$$

$$\tau^j \mapsto wj\eta \left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n.$$

In order to compute the Tate pairing of $\frac{1}{\eta}c^v_{v,s}$ and $\kappa_v$ we first must lift $\frac{1}{\eta}c^v_{v,s}$ to $H^1(F_v, T/l^nT)$. In fact, the same lifting

$$\theta^v : \mathrm{Gal}(K(\lambda_n)/F_v) \to T/l^nT$$

$$\varphi^i\tau^j \mapsto w\varepsilon(v)^i j \left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n$$

still works.

We now compute the cup product of $\kappa_v$ and $\theta^v$ as cohomology classes for $G$. Writing $C = \theta^v \cup \kappa_v$, one finds that

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \mathrm{tr}\Big(w\varepsilon(v)^i j\chi(v)^{-i'}\Big((-2bc_{i'} + ad_{i'} - dd_{i'})\omega(\partial a_{i'}) + (2cd_{i'} + ac_{i'} - dc_{i'})\omega(\partial b_{i'}) + (2ba_{i'} - ab_{i'} + db_{i'})\omega(\partial c_{i'}) + (-2cb_{i'} - aa_{i'} + da_{i'})\omega(\partial d_{i'})\Big)\Big)\zeta_n.$$

Applying Lemma 4.1, we find that this is simply

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = 2w\varepsilon(v)^i i' j\,\mathrm{tr}(\omega(\partial T_v))\zeta_n$$

and from here the computation is identical to the earlier case; we conclude that

$$\Xi(\partial T_v) = 2w\partial T_v.$$

### 6. Computation of $\Xi$ in the general case

We now remove the assumption that $\pi$ is an isomorphism. The computation in this case is essentially the same as in the previous case. First, recall that universality of $R$ means that, fixing a universal deformation $\rho_R$, there is some basis of $H$ with respect to which $\rho_A = \pi\rho_R$. We can conjugate in $\mathrm{GL}_2(A)$ from this basis to our fixed basis $x, y$; since $\pi$ is surjective (and $R$ is local) we can lift this conjugation to $\mathrm{GL}_2(R)$. That is, we can conjugate $\rho_R$ so as to assume that $\rho_A = \pi\rho_R$ where $\rho_A$ is now the representation on our fixed basis $x, y$ of $H$.

To compute $\Xi$ this time, we begin with

$$\omega : \Omega_R \otimes_R A \to A/l^n A$$

and compute its image in $\mathbf{Z}/l^n\mathbf{Z}$. Proceeding as before, we find that this is the image under the Tate pairing of two cocycles $\kappa_v : G \to T^*[l^n]$ and $\theta^v : G \to T/l^n T$. Writing

$$\rho_R(\mathrm{Fr}(v)^i) = \begin{pmatrix} \hat{a}_i & \hat{b}_i \\ \hat{c}_i & \hat{d}_i \end{pmatrix}$$

and

$$\rho_A(\mathrm{Fr}(v)^i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

we find that

$$\begin{aligned} &\kappa(\varphi^i\tau^j)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \\ &\quad = \mathrm{tr}\Big(\chi(v)^{-i}\big(\alpha d_i\omega(\partial\hat{a}_i) - \alpha b_i\omega(\partial\hat{c}_i) + \gamma d_i\omega(\partial\hat{b}_i) - \gamma b_i\omega(\partial\hat{d}_i) + \\ &\qquad\qquad \beta a_i\omega(\partial\hat{c}_i) - \beta c_i\omega(\partial\hat{c}_i) - \alpha a_i\omega(\partial\hat{d}_i) + \alpha c_i\omega(\partial\hat{b}_i)\big)\Big). \end{aligned}$$

The cocycle $\theta^v$ is given by

$$\theta^v(\varphi^i\tau^j) = w\varepsilon(v)^i j\left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n$$

where

$$\rho_A(\mathrm{Fr}(v)) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as before.

From these expressions the computation works out exactly as in the previous case, with $\partial a_i$, $\partial b_i$, $\partial c_i$, $\partial d_i$ replaced by $\partial\hat{a}_i$, $\partial\hat{b}_i$, $\partial\hat{c}_i$, $\partial\hat{d}_i$ respectively. Lemma 4.1 applies to show that

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = 2w\varepsilon(v)^i i' j\,\mathrm{tr}(\omega(\partial\hat{a} + \partial\hat{d}))\zeta_n,$$

where

$$\rho_R(\mathrm{Fr}(v)) = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}.$$

From here the computation is as before, with the fact that $\xi_7(\partial\hat{a}) = \partial a$ and $\xi_7(\partial\hat{d}) = \partial d$ showing that $\Xi$ is still multiplication by $2w$. This completes the proof of Theorem IV.6.2.

# Part 2

# Construction of cohesive Flach systems

CHAPTER 6

# The Flach map

In this chapter we define the Flach map from algebraic $K$-theory to Galois cohomology; this map will be used in Chapter IX to generate geometric Euler systems.

## 1. The coniveau spectral sequence in étale cohomology

We begin with the background material in algebraic geometry and algebraic $K$-theory which will be required for the remainder of this thesis. In particular, we will work in more generality then is actually needed for the definition of the Flach map.

We begin with a spectral sequence in étale cohomology which will be used in the construction of the Flach map. The main reference for this construction is [**Gro68**, Section 10.1]; see also [**CTHK97**, Section 1], [**Gil81**, pp. 239–242] and [**Fla95**, Section 5.1].

Let $X$ be a scheme of finite Krull dimension and let $\mathcal{F}$ be a torsion étale sheaf on $X$. Let $Y$ be a closed subscheme of $X$. For all $i, p$, define

$$\begin{aligned} H^i_Y(X,\mathcal{F})^p &= \varinjlim_{\substack{Z\subseteq Y\\ \operatorname{codim}_X Z\geq p}} H^i_Z(X,\mathcal{F}) \\ H^i_Y(X,\mathcal{F})^{p/p+1} &= \varinjlim_{\substack{Z'\subseteq Z\subseteq Y\\ \operatorname{codim}_X Z\geq p\\ \operatorname{codim}_X Z'\geq p+1}} H^i_{Z-Z'}(X-Z',\mathcal{F}). \end{aligned} \tag{1.1}$$

For each pair $Z' \subseteq Z$ as in (1.1) we have the usual exact sequence

$$\cdots \to H^i_{Z'}(X,\mathcal{F}) \to H^i_Z(X,\mathcal{F}) \to H^i_{Z-Z'}(X-Z',\mathcal{F}) \to H^{i+1}_{Z'}(X,\mathcal{F}) \to \cdots$$

Taking the direct limit over all such pairs (for fixed $p$) we obtain a long exact sequence

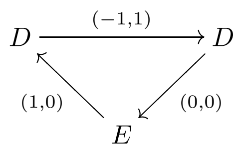
(1.2)
$$\cdots \to H^i_Y(X,\mathcal{F})^{p+1} \to H^i_Y(X,\mathcal{F})^p \to H^i_Y(X,\mathcal{F})^{p/p+1} \to H^{i+1}_Y(X,\mathcal{F})^{p+1} \to \cdots$$

Set

$$\begin{aligned} D &= \bigoplus_{p,q} H^{p+q}_Y(X,\mathcal{F})^p \\ E &= \bigoplus_{p,q} H^{p+q}_Y(X,\mathcal{F})^{p/p+1}. \end{aligned}$$

(1.2) yields an exact couple (in which we have labeled the maps by their $(p, q)$-bidegrees)

(1.3)




This in turn yields the *coniveau spectral sequence* (see [**Wei94**, Section 5.9] or Section A.8)

$$E_{1,Y}^{pq}(X, \mathcal{F}) = \bigoplus_{p,q} H_Y^{p+q}(X, \mathcal{F})^{p/p+1} \Rightarrow H_Y^{p+q}(X, \mathcal{F}). \tag{1.4}$$

Here $H_Y^n(X, \mathcal{F})$ appears as the direct limit of $H_Y^n(X, \mathcal{F})^p$ as $p$ goes to $-\infty$; in fact, it already appears at the $p = 0$ term.

We can compute the $E_1$-term of (1.4) a bit further. If $Z_1 \cap Z_2 = \emptyset$, then

$$H^i_{Z_1 \cup Z_2}(X, \mathcal{F}) \cong H^i_{Z_1}(X, \mathcal{F}) \oplus H^i_{Z_2}(X, \mathcal{F}), \tag{1.5}$$

as one sees easily from the definition of cohomology with support and excision. For the case where $Z_1$ and $Z_2$ are not necessarily disjoint closed subschemes of $X$, we can rewrite (1.5) as

$$\begin{aligned} H^i_{Z_1 \cup Z_2 - Z_1 \cap Z_2}&(X - Z_1 \cap Z_2, \mathcal{F}) \cong \\ &H^i_{Z_1 - Z_1 \cap Z_2}(X - Z_1 \cap Z_2, \mathcal{F}) \oplus H^i_{Z_2 - Z_1 \cap Z_2}(X - Z_1 \cap Z_2, \mathcal{F}). \end{aligned} \tag{1.6}$$

If $Z_1$ and $Z_2$ are also distinct, irreducible and of codimension $p$, then $Z_1 \cap Z_2$ has codimension at least $p + 1$ since $Z_1$ and $Z_2$ each have a unique generic point. For an arbitrary $Z$, splitting up each $H^{p+q}_{Z-Z'}(X - Z', \mathcal{F})$ into the pieces corresponding to the irreducible components of $Z$ and using (1.6) we find that

$$\begin{aligned} H^i_Y(X, \mathcal{F})^{p/p+1} &\overset{\text{def}}{=} \varinjlim_{\substack{Z' \subseteq Z \subseteq Y \\ \operatorname{codim}_X Z \geq p \\ \operatorname{codim}_X Z' \geq p+1}} H^i_{Z-Z'}(X - Z', \mathcal{F}) \\ &\cong \bigoplus_{x \in X^p \cap Y} \varinjlim_{Z' \subsetneq \bar{x}} H^i_{\bar{x}-Z'}(X - Z', \mathcal{F}). \end{aligned} \tag{1.7}$$

Here by $\bar{x}$ we mean the closure of $\{x\} \subseteq X$ regarded as a reduced closed subscheme of $X$.

For each $x \in X^p$ we take this last expression as the definition of $H^i_x(X, \mathcal{F})$; that is,

$$H^i_x(X, \mathcal{F}) \overset{\text{def}}{=} \varinjlim_{Z \subsetneq \bar{x}} H^i_{\bar{x}-Z}(X - Z, \mathcal{F}). \tag{1.8}$$

These groups are easily seen to be contravariant for flat morphisms in the following sense: if $f : X' \to X$ is a flat morphism, then for each $x \in X^p$ there is a map

$$H^i_x(X, \mathcal{F}) \to \bigoplus_{x'_i} H^i_{x'_i}(X', f^*\mathcal{F});$$

here the $x'_i$ are the generic points of the irreducible components of $f^{-1}(\bar{x})$ of codimension $p$; such points will exist by [**GDb**, Corollary 6.1.4]. The maps to the $H^i_{x'}(X', f^*\mathcal{F})$ for $x'$ the generic point of an irreducible component of $f^{-1}(\bar{x})$ of codimension greater than $p$ will all be zero, as the closed set $Z$ in (1.8) can be taken so that $f^{-1}(Z)$ contains $x'$.

The cohomology groups (1.8) are also covariant for finite flat morphisms: if $f : X' \to X$ is a finite flat morphism, then for each $x' \in X'^p$ there is a map

$$H^i_{x'}(X', f^*\mathcal{F}) \to H^i_{f(x')}(X, \mathcal{F})$$

induced by the trace map in étale cohomology; see [**FK88**, pp. 133-135] for the definition of the trace map. Note that $\operatorname{codim} f(x') = p$ since $f$ is finite flat. Summing up, we have the following proposition.

PROPOSITION 1.1. *Let $X$ be a scheme of finite Krull dimension, let $Y$ be a closed subscheme of $X$ and let $\mathcal{F}$ be a torsion sheaf on $X$. Then there is a spectral sequence*

$$E^{pq}_{1,Y}(X, \mathcal{F}) = \bigoplus_{x \in X^p \cap Y} H^{p+q}_x(X, \mathcal{F}) \Rightarrow H^{p+q}_Y(X, \mathcal{F}). \tag{1.9}$$

*If $f : X' \to X$ is a flat morphism and $Y'$ is a closed subscheme of $X'$ containing $f^{-1}(Y)$, then there is an induced morphism of spectral sequences*

$$E^{pq}_{r,Y}(X, \mathcal{F}) \to E^{pq}_{r,Y'}(X', f^*\mathcal{F})$$

*and the map on $E_1$-terms is the same as that coming from the contravariant functoriality of $H^*_x$. Furthermore, if for some $p$, $q$ and $r$ there are edge maps (see Section A.2)*

$$E^{pq}_{r,Y}(X, \mathcal{F}) \to H^{p+q}_Y(X, \mathcal{F})$$
$$E^{pq}_{r,Y'}(X', f^*\mathcal{F}) \to H^{p+q}_{Y'}(X', f^*\mathcal{F}),$$

*then the diagram*

$$\begin{array}{ccc} E^{pq}_{r,Y}(X, \mathcal{F}) & \longrightarrow & E^{pq}_{r,Y'}(X', f^*\mathcal{F}) \\ \downarrow & & \downarrow \\ H^{p+q}_Y(X, \mathcal{F}) & \longrightarrow & H^{p+q}_{Y'}(X', f^*\mathcal{F}) \end{array}$$

*commutes. The corresponding statements remain true for covariant functoriality with respect to finite flat morphisms, with the obvious modifications.*

PROOF. The existence of the spectral sequence and the expression (1.9) were proven above. To see the functoriality, note first that $f$ induces maps of the long exact localization sequences, and thus of the exact couples (1.3) for the pairs $X, Y$ and $X', Y'$. (Note that flatness is needed here to insure that the relevant codimensions are compatible.) This yields the map of spectral sequences (see Section A.8), and the fact that for $r = 1$ this map is the same as that coming from the maps on the $H^i_x(X, \mathcal{F})$'s is immediate from the functoriality of the isomorphisms (1.7) The compatibility of the edge maps is proven in Proposition A.8.1. The same arguments work for the covariant case. □

## 2. The localization sequence

We continue with the notation of the previous section. Write $U$ for the open subscheme $X - Y$ of $X$. We will write $E^{pq}_r(X, \mathcal{F})$ for the spectral sequence previously denoted $E^{pq}_{r,X}(X, \mathcal{F})$. Contravariant functoriality for flat morphisms and covariant functoriality for finite flat both yield natural maps

$$E^{pq}_{r,Y}(X, \mathcal{F}) \to E^{pq}_r(X, \mathcal{F})$$

arising from the identity map on $X$; these maps coincide. These maps are also compatible with either of the expressions (1.4) and (1.9). By contravariance for flat morphisms we also have a map

$$E_r^{pq}(X,\mathcal{F}) \to E_r^{pq}(U,\mathcal{F}).$$

LEMMA 2.1. *Assume that $U$ is dense in $X$. For each $q$, there is a short exact sequence of complexes*

$$0 \to E_{1,Y}^{\bullet q}(X,\mathcal{F}) \to E_1^{\bullet q}(X,\mathcal{F}) \to E_1^{\bullet q}(U,\mathcal{F}) \to 0. \tag{2.1}$$

*This induces a long exact sequence*

$$\cdots \to E_{2,Y}^{pq}(X,\mathcal{F}) \to E_2^{pq}(X,\mathcal{F}) \to E_2^{pq}(U,\mathcal{F}) \to E_{2,Y}^{p+1,q}(X,\mathcal{F}) \to \cdots \tag{2.2}$$

*Furthermore, suppose that for some $p,q$ there exist edge maps forming a square with the boundary maps of (2.2):*

$$\begin{array}{ccc} E_2^{pq}(U,\mathcal{F}) & \longrightarrow & E_{2,Y}^{p+1,q}(X,\mathcal{F}) \\ \downarrow & & \downarrow \\ H^{p+q}(U,\mathcal{F}) & \longrightarrow & H_Y^{p+q+1}(X,\mathcal{F}) \end{array}$$

*Then this square commutes.*

PROOF. We have already seen the existence of the maps of spectral sequences in (2.1). Note that the differentials at this stage are vertical, so that for fixed $q$ we can consider $E_1^{\bullet q}$ as a complex. The fact that the maps are maps of spectral sequences immediately implies that (2.1) is a maps of complexes. To construct (2.2) we therefore must show only that (2.1) is exact as a sequence of abelian groups.

Since $U$ is dense in $X$, $U^p = X^p \cap U$. The map

$$E_1^{pq}(X,\mathcal{F}) \to E_1^{pq}(U,\mathcal{F})$$

is the direct sum of the maps

$$H_x^{p+q}(X,\mathcal{F}) \to H_x^{p+q}(U,\mathcal{F}) \tag{2.3}$$

over all $x \in U^p$. In particular, for $x \notin U$ the terms $H_x^{p+q}(X,\mathcal{F})$ map to zero in $E_1^{pq}(U,\mathcal{F})$; thus by (1.9) $E_{1,Y}^{pq}(X,\mathcal{F})$ is in the kernel of the map $E_1^{pq}(X,\mathcal{F}) \to E_1^{pq}(U,\mathcal{F})$.

Fix now one $x \in U^p$. Plugging back into the definitions, in (2.3) we are considering the direct limit over $Z \subsetneq \bar{x}$ of the natural maps

$$H_{\bar{x}-Z}^{p+q}(X-Z,\mathcal{F}) \to H_{\bar{x}\cap U - Z\cap U}^{p+q}(U - Z\cap U,\mathcal{F}). \tag{2.4}$$

For any $Z$ containing $Y \cap \bar{x}$ the map (2.4) is an isomorphism by excision; since such $Z$ are cofinal in the set of all $Z$, the direct limit of the maps (2.4) is an isomorphism. It is now clear that the kernel of the map $E_1^{pq}(X,\mathcal{F}) \to E_1^{pq}(U,\mathcal{F})$ is precisely $E_{1,Y}^{pq}(X,\mathcal{F})$, so that (2.1) is exact as a sequence of abelian groups.

The exact sequence (2.1) yields (2.2) in the usual way. For the compatibility of the boundary maps with the edge maps, see [**Fla95**, Proposition 3]. □

## 3. Grothendieck's purity conjecture

We will need Grothendieck's purity conjecture in order to get an additional expression for our coniveau spectral sequence. We begin by briefly recalling the statement of the conjecture as we will need it; see [**Gro77**, Exposé 1, Section 3.1.4], [**CT95**, Section 3.2] or [**?**, Section 1] for more details. For a closed immersion $i : Y \hookrightarrow X$ and a sheaf $\mathcal{F}$ on $X$, we write $i^!\mathcal{F}$ for the sheaf of sections of $Y$ supported on $Y$; see [**FK88**, Chapter I, Section 10].

Conjecture 3.1. *Let $X$ be a regular scheme and let $i : Y \to X$ be a closed immersion of a regular scheme $Y$. Assume that $i$ has codimension $c$ at every point. Let $\mathcal{F}$ be a locally constant torsion sheaf on $X$ of exponent invertible in $\mathcal{O}_X$. Then there are functorial isomorphisms*

$$R^j i^! \mathcal{F} \cong \begin{cases} 0 & j \neq 2c; \\ i^*\mathcal{F}(-c) & j = 2c. \end{cases}$$

*In particular,*

$$H_Y^{j+2c}(X, \mathcal{F}) \cong H^j\big(Y, i^*\mathcal{F}(-c)\big)$$

*for all $j$.*

We will need the following results on the purity conjecture.

Theorem 3.2. *Let $X$ be a regular scheme and let $Y$ be an irreducible regular closed subscheme. Then the purity conjecture is known for the inclusion $Y \hookrightarrow X$ in any of the following circumstances:*

1. *$X$ and $Y$ are both smooth over a base $S$;*
2. *$X$ is a scheme of finite type over a perfect field;*
3. *$X$ is a smooth scheme over a discrete valuation ring with perfect residue field and $Y$ is a closed subscheme of the special fiber of $X$;*
4. *$X$ is a separated scheme of finite type over a local or global field of positive characteristic and the sheaf $\mathcal{F}$ has exponent divisible only by primes $\geq \dim X + 2$.*

Proof. (1) is the usual purity theorem in étale cohomology; see [**GAV73**, Exposé 16, Section 3] or [**FK88**, Chapter I, Theorem 10.1]. (2) follows from this and the fact [**GDb**, Proposition 17.15.1] that regular and smooth are the same for schemes of finite type over a perfect field. The case of (3) where $Y$ is the special fiber of $X$ is [**Ras89**, Lemma 2.1]; the general case follows easily from the long exact sequence of a pair and (2). (4) is proved in [**?**, Corollary 3.7], together with the cohomological dimension calculations of [**GAV73**, Exposé 10, Theorem 4.3 and Theorem 5.2] and [**Ser97**, Corollary of Section II.4.2 and Proposition 12]. □

If $Y$ is a closed subscheme of $X$, we will say that the pair $X, Y$ satisfies *relative purity at $N$* if for all irreducible regular closed subschemes $Z$ of $Y$ (of pure codimension in $X$) and all locally constant $N$-torsion sheaves $\mathcal{F}$ on $X$, the purity conjecture is satisfied for the inclusion of $Z$ into $X$. We will say that $X$ satisfies *purity at $N$* if the pair $X, X$ satisfies relative purity at $N$.

Now let $X$ be a regular scheme of finite type over a field or a discrete valuation ring. Let $Y$ be a closed subscheme. Assume that relative purity at $N$ holds for the pair $X, Y$. In this situation we can further simplify the coniveau spectral sequence. Again, the main reference is [**Gro68**, Section 10.1] (we should note that in his treatment he seems to be assuming that the base field is perfect); see also [**CTHK97**, Section 1].

Let $x \in Y$ be of codimension $p$ in $X$, let $\mathcal{F}$ be a locally constant $N$-torsion sheaf on $X$ and consider

$$H^i_x(X, \mathcal{F}) = \varinjlim_{Z \subsetneq \bar{x}} H^i_{\bar{x}-Z}(X - Z, \mathcal{F}).$$

Since $X$ is assumed to be regular, $X - Z$ is certainly regular for each $Z$. Furthermore, $\bar{x}$ is generically regular since the local ring of the generic point is the field $k(x)$. Together with [**GDb**, Corollary 6.12.6] this implies that $\bar{x}$ is regular on a non-empty open subscheme. It follows that the regular open sets of $\bar{x}$ are cofinal among all open sets of $\bar{x}$, and thus that

$$H^i_x(X, \mathcal{F}) = \varinjlim_{\substack{Z \subsetneq \bar{x} \\ \bar{x}-Z \text{ regular}}} H^i_{\bar{x}-Z}(X - Z, \mathcal{F}).$$

Purity tells us that for any such $Z$ we have

$$H^i_{\bar{x}-Z}(X - Z, \mathcal{F}) \cong H^{i-2p}\big(\bar{x} - Z, \mathcal{F}(-p)\big).$$

Thus

$$H^i_x(X, \mathcal{F}) \cong \varinjlim_{\substack{Z \subsetneq \bar{x} \\ X-Z \text{ regular}}} H^{i-2p}\big(\bar{x} - Z, \mathcal{F}(-p)\big).$$

We can further restrict the direct system to affine open sets in $\bar{x}$, as they are a base for the topology on $\bar{x}$. In this situation étale cohomology commutes with the direct limit (see [**GAV73**, Exposé 7, Section 5.8] or [**Art62**, Chapter 1, III.3]), so that

$$H^i_x(X, \mathcal{F}) \cong H^{i-2p}\big(\varprojlim\, (\bar{x} - Z), \mathcal{F}(-p)\big).$$

But this inverse limit is simply $\operatorname{Spec} k(x)$ (using here again the fact that $\bar{x}$ is reduced), so we conclude finally that

$$H^i_x(X, \mathcal{F}) \cong H^{i-2p}\big(\operatorname{Spec} k(x), \mathcal{F}(-p)\big).$$

We summarize this in a proposition.

Proposition 3.3. *Let $X$ be a regular scheme of finite type over a field or discrete valuation ring; let $Y$ be a closed subscheme of $X$ such that relative purity at $N$ holds for the pair $X, Y$. Then the $E_1$-term of the coniveau spectral sequence can be written as*

$$E^{pq}_{1,Y}(X, \mathcal{F}) \cong \bigoplus_{x \in X^p \cap Y} H^{q-p}\big(\operatorname{Spec} k(x), \mathcal{F}(-p)\big)$$

*and this identification respects the functorialities on both sides.*

Proof. The only new statement is the last one, and this follows from the functoriality assumptions in the purity conjecture. □

## 4. The coniveau spectral sequence in $K$-theory

One can redo the entire construction of the previous three sections using algebraic $K$-theory rather than étale cohomology. This construction is carried out in [**Fla95**, Sections 5.1 and 5.2]; for regular schemes it is equivalent to the construction in [**Qui73**, Section 7, Theorem 5.4]. We will also need the third equivalent construction given in [**Gil81**, pp. 239-240 and pp. 271-272]. For later reference we state what we will need as a proposition. Following [**Gil81**, Definition 2.13], if $Y$

is a closed subscheme of $X$, we define the *relative $K$-groups* $K_{i,Y}(X)$ to be the homotopy fibers of $K(X) \to K(X-Y)$. If $X$ is a scheme of finite Krull dimension, we define the *codimension $p$ Chow group* $A^pX$ to be the cokernel of the map

$$\bigoplus_{x\in X^{p-1}} k(x)^\times \to \bigoplus_{x\in X^p} \mathbf{Z}$$

where the map

$$k(x)^\times \to \bigoplus_{x'\in\bar{x}^1} \mathbf{Z} \tag{4.1}$$

sends a rational function $f$ to its divisor in the sense of [**Ful98**, Section 1.3]; note that the definition there works perfectly well for schemes with are not finite type over a field. See, for example, [**?**, Chapter 1].

PROPOSITION 4.1. *Let $X$ be a regular noetherian scheme of finite Krull dimension and let $Y$ be a closed subscheme of $X$. Then there is a spectral sequence*

$$E_{1,Y}^{pq}(X) = \bigoplus_{x\in X^p\cap Y} K_{-p-q}k(x) \Rightarrow K_{-p-q,Y}X. \tag{4.2}$$

*This spectral sequence is contravariant for flat morphisms, covariant for finite flat morphisms, and these functorialities are compatible with edge maps. If $U = X - Y$, then there is a localization sequence as in Lemma 2.1. Finally, for any $p$ the spectral sequence differential*

$$\begin{array}{ccc} E_{1,Y}^{p-1,-p}(X) & \longrightarrow & E_{1,Y}^{p,-p}(X) \\ \| & & \| \\ \bigoplus_{x\in X^{p-1}\cap Y} k(x)^\times & & \bigoplus_{x\in X^p\cap Y} \mathbf{Z} \end{array}$$

*identifies with the direct sum of the maps (4.1). In particular, $E_2^{p,-p}(X)$ identifies with the codimension $p$ Chow group $A^pX$.*

PROOF. The spectral sequence (4.2) is initially constructed in [**Qui73**, Section7, Theorem 5.4], where he also proves contravariant functoriality for flat morphisms. Covariant functoriality is proven in the same way, using [**Qui73**, Section 7, (2.7) and (2.8)]. The compatibility with edge maps is proven using Flach's construction and the methods of the previous sections of this chapter. The localization sequence arises in the same way. Finally, the differential computation is in [**Qui73**, Proposition 5.14 and Remark 5.17], together with [**Gra77**]; note that Quillen's statement is somewhat awkward, but in the proof it is apparent that he is proving precisely the statement above. □

We will write a general element of $E_{1,Y}^{pq}(X)$ as $\sum(\alpha_i, f_i)$ where $\alpha_i \in X^p \cap Y$ and $f_i \in K_{-p-q}k(\alpha_i)$. More generally, let $\alpha$ be a closed subscheme of $Y$ such that each irreducible component $\alpha_i$ has codimension $p$ in $X$. If $f$ is a section of the Zariski sheaf $\mathcal{K}_{-p-q}\alpha$ defined on a dense open set, then we write $(\alpha, f)$ for the element $\sum(\alpha_i, f_i^{m_i})$ of $E_{1,Y}^{pq}(X)$; here $f_i$ is the restriction of $f$ to $\alpha_i$ and $m_i$ is the length of the local ring of $\alpha_i$ in $\alpha$.

The coniveau spectral sequences in $K$-theory and étale cohomology are connected by Chern class maps constructed by Gillet in [**Gil81**]; see also [**Lev98**, Chapter 3]. We summarize what we will need in another proposition.

PROPOSITION 4.2. *Let $X$ be a regular noetherian scheme of finite Krull dimension, let $Y$ be a closed subscheme of $X$ and let $\mathcal{F}$ be the sheaf $\mathbf{Z}/N\mathbf{Z}$ for some $N$ which is invertible on $X$. Then for any $i$ and $j$ there is a natural transformation of functors*

$$K_{i,Y}X \to H_Y^{2i-j}\big(X, \mathcal{F}(i)\big). \tag{4.3}$$

*These combine to give a map of coniveau spectral sequences*

$$E_{r,Y}^{pq}(X) \to E_{r,Y}^{p,q+2i}\big(X, \mathcal{F}(i)\big) \tag{4.4}$$

*which is functorial in $X$ and $Y$ (both contravariantly for flat morphisms and covariantly for finite flat morphisms). These maps also commute with the respective localization sequences. Lastly, assuming also that $X$ is of finite type over a field or a discrete valuation ring and that the pair $X, Y$ satisfies relative purity at $N$, the map*

$$\begin{array}{ccc} E_{r,Y}^{p,-p}(X) & \longrightarrow & E_{r,Y}^{p,p}\big(X, \mathcal{F}(p)\big) \\ \Big\| & & \Big\| \\ \bigoplus_{x \in X^p \cap Y} \mathbf{Z} & \longrightarrow & \bigoplus_{x \in X^p \cap Y} H^0(\operatorname{Spec} k(x), \mathcal{F}) \end{array} \tag{4.5}$$

*is just the direct sum of the canonical maps $\mathbf{Z} \to H^0(\operatorname{Spec} k(x), \mathcal{F})$ for the constant sheaf $\mathcal{F}$.*

PROOF. See [**Gil81**, esp. Definition 2.22 and Lemma 2.23] and [**Lev98**, Chapter 3, Section 1.4] for the construction of (4.3). The corresponding maps (4.4) are given in [**Gil81**, pp. 239-240] and the localization behavior is in [**Lev98**, Chapter 3, Section 1.5]. (4.5) follows easily from Riemann-Roch without denominators as in [**Lev98**, Theorem 3.4.7], using the methods outlined in [**Gil81**, Theorem 3.9 and Remark 3.10] together with our purity hypotheses. (Note that [**Gil81**, Theorem 3.9] does not actually apply in this situation.) □

## 5. Definition of the Flach map

We are now in a position to define the Flach map. Let $F$ be a field and let $X$ be a smooth separated $F$-scheme of finite type and dimension $n$. Let $N$ be an integer relatively prime to the characteristic of $F$ and let $\mathcal{F}$ be the constant sheaf $\mathbf{Z}/N\mathbf{Z}$ on $X$. Fix also an integer $m$, $0 \leq m \leq n$. To define the Flach map we need to assume that $X$ satisfies purity at $N$. By Theorem 3.2 this assumption holds if $F$ is a perfect field or if $F$ is a local or global field of positive characteristic and $N$ is divisible only by primes $\geq n+2$.

We will work with the second stage of the coniveau spectral sequence for reasons which will become clear in a moment. We begin with the Chern class map (4.4) (with $p = m$, $q = -m-1$, $i = m+1$ and $Y = X$), which we denote by $c(X)$:

$$E_2^{m,-m-1}(X) \xrightarrow{c(X)} E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big).$$

Since we assumed that purity holds for $X$ at $N$ by Proposition 3.3, we can write the $E_1$-term of this second spectral sequence as

$$E_1^{pq}\big(X, \mathcal{F}(m+1)\big) = \bigoplus_{x \in X^p} H^{q-p}\big(\operatorname{Spec} k(x), \mathcal{F}(m+1-p)\big).$$

But this certainly vanishes for $q < p$; it follows that there is an edge map (see Example A.2.1) at the second stage:

$$E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big) \xrightarrow{d(X)} H^{2m+1}\big(X, \mathcal{F}(m+1)\big).$$

Note that this edge map does not yet exist at the first stage.

We now use the Leray spectral sequence [**FK88**, p. 28] for the morphism $u : X \to \operatorname{Spec} F$:

$$H^p\big(\operatorname{Spec} F, R^q u_* \mathcal{F}(m+1)\big) \Rightarrow H^{p+q}\big(X, \mathcal{F}(m+1)\big). \tag{5.1}$$

(5.1) yields an edge map

$$H^{2m+1}\big(X, \mathcal{F}(m+1)\big) \to H^0\big(F, R^{2m+1} u_* \mathcal{F}(m+1)\big). \tag{5.2}$$

Define $H^{2m+1}(X, \mathcal{F}(m+1))_0$ to be the kernel of (5.2); (5.1) yields a natural edge map

$$H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0 \xrightarrow{e(X)} H^1\big(F, R^{2m} u_* \mathcal{F}(m+1)\big)$$

Often the 0-part fills up the entire étale cohomology group.

Lemma 5.1. *Assume that $H^{2m+1}(X_{F_s}, \mathcal{F}(m+1))$ has no $G_F$-invariants. Then $H^{2m+1}(X, \mathcal{F}(m+1))_0 = H^{2m+1}(X, \mathcal{F}(m+1))$.*

Proof. Recall that étale sheaves on $\operatorname{Spec} F$ can be identified with discrete $G_F$-modules and that under this identification étale cohomology identifies with Galois cohomology. The sheaf $R^{2m+1}u_*\mathcal{F}(m+1)$ corresponds to the Galois module $H^{2m+1}(X_{F_s}, \mathcal{F}(m+1))$; thus by the assumption of the lemma

$$H^0\big(\operatorname{Spec} F, R^{2m+1} u_* \mathcal{F}(m+1)\big) = H^0\big(F, H^{2m+1}(X_{F_s}, \mathcal{F}(m+1))\big) = 0.$$

This proves the lemma. □

Set

$$E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big)_0 = d(X)^{-1} H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0$$
$$E_2^{m,-m-1}(X)_{0,\mathcal{F}} = c(X)^{-1} d(X)^{-1} H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0.$$

Definition 5.2. The Flach map

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}} \to H^1\big(\operatorname{Spec} F, R^{2m} u_* \mathcal{F}(m+1)\big)$$

is defined to be to be $e(X) \circ d(X) \circ c(X)$.

We can also consider the Flach map as a map to Galois cohomology. The étale sheaf $R^{2m}u_*\mathcal{F}(m+1)$ identifies with the $G_F$-module $H^{2m}(X_{F_s}, \mathcal{F}(m+1))$; denote this $G_F$-module as $V$. Under these identifications, étale cohomology becomes Galois cohomology, so we can write the Flach map as

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}} \to H^1(F, V).$$

The domain of $\sigma_m$ is not a particularly complicated object. Indeed, the group $E_2^{m,-m-1}(X)$ is just the cohomology of the complex

$$\bigoplus_{x \in X^{m-1}} K_2 k(x) \to \bigoplus_{x \in X^m} k(x)^\times \to \bigoplus_{x \in X^{m+1}} \mathbf{Z}$$

where the second map is the divisor map. Thus $E_2^{m,-m-1}(X)$ identifies with a quotient of

$$\ker\left(\bigoplus_{x \in X^m} k(x)^\times \to \bigoplus_{x \in X^{m+1}} \mathbf{Z}\right); \tag{5.3}$$

this description makes it significantly simpler to exhibit elements of $E_2^{m,-m-1}(X)$.

## 6. Functoriality and passage to the limit

We keep the hypotheses of the previous section. Let $\mathcal{F}'$ be the constant sheaf $\mathbf{Z}/N'\mathbf{Z}$ on $X$ for some $N'$ dividing $N$. Let $\pi : \mathcal{F} \to \mathcal{F}'$ be the natural map. We have already assumed that $X$ satisfies purity at $N$, so there are two Flach maps

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}} \to H^1(F,V)$$

$$\sigma'_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}'} \to H^1(F,V')$$

where $V' = H^{2m}(X_{F_s}, \mathcal{F}'(m+1))$. We claim that these two maps are compatible, in the sense that $\sigma'_m$ is the composition of $\sigma_m$ with the natural map

$$H^1(F,V) \to H^1(F,V')$$

coming from $\pi$.

To check this compatibility we must check the commutativity of the diagram

$$\begin{array}{ccc}
E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \longrightarrow & E_2^{m,-m-1}(X)_{0,\mathcal{F}'} \\
\downarrow{\scriptstyle c(X)} & & \downarrow{\scriptstyle c'(X)} \\
E_2^{m,m+1}\big(X,\mathcal{F}(m+1)\big)_0 & \xrightarrow{\pi_1} & E_2^{m,m+1}\big(X,\mathcal{F}'(m+1)\big)_0 \\
\downarrow{\scriptstyle d(X)} & & \downarrow{\scriptstyle d'(X)} \\
H^{2m+1}\big(X,\mathcal{F}(m+1)\big)_0 & \xrightarrow{\pi_2} & H^{2m+1}\big(X,\mathcal{F}'(m+1)\big)_0 \\
\downarrow{\scriptstyle e(X)} & & \downarrow{\scriptstyle e'(X)} \\
H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big) & \xrightarrow{\pi_3} & H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}'(m+1)\big)
\end{array}$$

where the $\pi_i$ are the maps induced by $\pi$. (The fact that these maps send 0-parts to 0-parts is immediate.) This is quite easy: the commutativity of the first square follows from the functoriality of the Chern class maps; the commutativity of the second square is proven in Proposition A.8.1; and the commutativity of the third square is Proposition A.4.1.

Fix now a prime $l$ such that $X$ satisfies purity at all powers of $l$. Considering $\mathbf{Z}/l^i\mathbf{Z}$ as a $\mathbf{Z}/l^{i+1}\mathbf{Z}$-module, the compatibility above shows that the Flach maps are compatible for the constant sheaves associated to $\mathbf{Z}/l^i\mathbf{Z}$ for all $i$. We will use this to define a Flach map for the sheaf $\mathbf{Z}_l$. Let $E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l}$ be the set of all elements of $E_2^{m,-m-1}(X)$ which lie in $E_2^{m,-m-1}(X)_{0,\mathbf{Z}/l^i\mathbf{Z}}$ for some (and thus all sufficiently large) $i$. Set

$$V = H^{2m}\big(X_{F_s}, \mathbf{Z}_l(m+1)\big) \overset{\text{def}}{=} \varprojlim_i H^{2m}\big(X_{F_s}, \mathbf{Z}/l^i\mathbf{Z}(m+1)\big).$$

Passing to the limit we obtain a Flach map

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \to H^1(F,V).$$

This is the version of the Flach map which we will almost always consider in later chapters. The next two results show that we can often consider this $\sigma_m$ as originating in $E_2^{m,-m-1}(X)$.

LEMMA 6.1. *Let $\mathfrak{X}$ be a smooth and projective scheme over the ring of integers of a local field $K$. Let $l$ be relatively prime to the residue characteristic of $K$; if $K$ has positive characteristic then assume that $l \geq n+2$. Set $X = \mathfrak{X}_K$. Then*

$$E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \otimes_{\mathbf{Z}} \mathbf{Q} = E_2^{m,-m-1}(X) \otimes_{\mathbf{Z}} \mathbf{Q}. \tag{6.1}$$

*If $H^{2m+1}(X_{K_s}, \mathbf{Z}_l)$ is torsion-free, then*

$$E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} = E_2^{m,-m-1}(X). \tag{6.2}$$

PROOF. Let $k$ denote the residue field of $K$. Smooth base change and the Weil conjectures [**FK88**, Chapter IV, Theorem 1.2] show that $H^{2m+1}(X_{K_s}, \mathbf{Q}_l(m+1))$ is unramified and that the eigenvalues of a geometric Frobenius on it are $\sqrt{\#k}$. Thus this étale cohomology group has trivial $G_K$-invariants. By Lemma 5.1 this implies that

$$H^{2m+1}\big(X, \mathbf{Z}_l(m+1)\big)_0 \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = H^{2m+1}\big(X, \mathbf{Z}_l(m+1)\big) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l.$$

(6.1) now follows since any element of $H^{2m+1}(X, \mathbf{Z}_l(m+1))$ which is not $G_K$-invariant yields elements of $H^{2m+1}(X, \mathbf{Z}_l/l^i\mathbf{Z}(m+1))$ which are not $G_K$-invariant for all sufficiently large $i$. (6.2) is proven in the same way, since the torsion-free hypothesis implies that $H^{2m+1}(X_{K_s}, \mathbf{Z}_l(m+1))$ itself already has no $G_K$-invariants. □

LEMMA 6.2. *Let $X$ be a smooth and projective scheme over a global field $F$. Let $l$ be relatively prime to the characteristic of $F$, and assume that $l \geq n+2$ if $F$ has positive characteristic. Then*

$$E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \otimes_{\mathbf{Z}} \mathbf{Q} = E_2^{m,-m-1}(X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

*If $H^{2m+1}(X_{F_s}, \mathbf{Z}_l)$ is torsion-free, then*

$$E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} = E_2^{m,-m-1}(X).$$

PROOF. Standard arguments show that $X$ extends to a smooth projective scheme over an open subscheme $S$ of the spectrum of the ring of integers of $F$. Applying Lemma 6.1 to the completion of $F$ at any closed point of $S$ of residue characteristic different from $l$ then proves the result. □

## 7. Functoriality II

Let $X$ and $F$ be as before. Let $F'$ be a field and let $X'$ be a smooth separated $F'$-scheme of finite type and the same dimension $n$. Suppose that we are given an inclusion of fields $F \hookrightarrow F'$ and a flat morphism of schemes $f : X' \to X$ compatible with this inclusion. Suppose also that $X'$ satisfies purity at $N$, so that there is a Flach map $\sigma'_m : E_2^{m,-m-1}(X')_0 \to H^1(F', V')$ where $V' = H^{2m}(X'_{F'_s}, f^*\mathcal{F}(m+1))$. We claim that $\sigma'_m$ is compatible with $\sigma_m$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \longrightarrow & E_2^{m,-m-1}(X')_{0,f^*\mathcal{F}} \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle \sigma'_m} \\ H^1(F, V) & \longrightarrow & H^1(F', V') \end{array}$$

Here the bottom map is the composition

$$H^1(F, V) \to H^1(F', V) \to H^1(F', V')$$

induced by the map of Galois groups $G_{F'} \to G_F$ and the map on étale cohomology $V \to V'$.

The proof of this is straightforward: we must check that the diagram

$$
\begin{array}{ccc}
E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{f_1} & E_2^{m,-m-1}(X')_{0,f^*\mathcal{F}} \\
\downarrow{\scriptstyle c(X)} & & \downarrow{\scriptstyle c(X')} \\
E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{f_2} & E_2^{m,m+1}\big(X', f^*\mathcal{F}(m+1)\big)_0 \\
\downarrow{\scriptstyle d(X)} & & \downarrow{\scriptstyle d(X')} \\
H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{f_3} & H^{2m+1}\big(X', f^*\mathcal{F}(m+1)\big)_0 \\
\downarrow{\scriptstyle e(X)} & & \downarrow{\scriptstyle e(X')} \\
H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big) & \xrightarrow{f_4} & H^1\big(\operatorname{Spec} F', R^{2m}u'_*f^*\mathcal{F}(m+1)\big)
\end{array}
\tag{7.1}
$$

is commutative. Here $u' : X' \to \operatorname{Spec} F'$ is the structure map and the $f_i$ are the obvious maps, which we discuss in more detail now.

$f_1$ is the map of coniveau spectral sequences induced by the flat morphism $f : X' \to X$. $f_2$ is induced by the same morphism, and this square commutes since Gillet's construction is a natural transformation.

$f_3$ is the map on étale cohomology coming from the morphism $X' \to X$; the second square commutes by Proposition 1.1.

For the fourth horizontal map, consider first the base change map

$$
f'^* R^{2m} u_* \mathcal{F} \to R^{2m} u'_* f^* \mathcal{F} \tag{7.2}
$$

coming from the cartesian square

$$
\begin{array}{ccc}
X & \xleftarrow{f} & X' \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} \\
\operatorname{Spec} F & \xleftarrow{f'} & \operatorname{Spec} F'
\end{array}
$$

where $f'$ is the obvious map. (7.2) induces a map

$$
H^1(\operatorname{Spec} F', f'^* R^{2m} u_* \mathcal{F}) \to H^1(\operatorname{Spec} F', R^{2m} u'_* f^* \mathcal{F}).
$$

Precomposing this with the map

$$
H^1(\operatorname{Spec} F, R^{2m} u_* \mathcal{F}) \to H^1(\operatorname{Spec} F', f'^* R^{2m} u_* \mathcal{F})
$$

coming from the morphism $f'$ defines $f_4$. That this square commutes is a standard result on the Leray spectral sequence and edge maps; see Proposition A.5.1. This completes the proof of the functoriality of the Flach map for flat morphisms as above.

There are two special cases of the above construction which are especially important: the first is when $F = F'$, so that $X' \to X$ is just a flat morphism of relative dimension 0 of $F$-schemes. The second is when $X' = X_{F'}$ is just the base change of $X$ and the morphism $X' \to X$ is the projection.

The last functoriality we will need is for finite flat morphisms. Let $f : X' \to X$ be a finite flat morphism of smooth separated $F$-schemes of dimension $n$ and suppose that all of the relevant Flach hypotheses are satisfied. Then there is a commutative diagram

$$\begin{array}{ccc} E_2^{m,-m-1}(X')_{0,\mathcal{F}} & \longrightarrow & E_2^{m,-m-1}(X)_{0,f^*\mathcal{F}} \\ \downarrow{\scriptstyle \sigma'_m} & & \downarrow{\scriptstyle \sigma_m} \\ H^1(F,V') & \longrightarrow & H^1(F,V) \end{array}$$

where $V' = H^{2m}(X'_F, f^*\mathcal{F}(m+1))$ and the horizontal maps come from covariant functoriality for finite flat morphisms. The proof of this is virtually the same as the proof above, given that all of our constructions were functorial both for flat and for finite flat morphisms.

CHAPTER 7

# Local analysis of the Flach map

The usefulness of the Flach map in generating geometric Euler systems comes from a geometric description of the local ramification of the image of the Flach map. The formulation and proof of this description is the focus of this chapter.

## 1. Overview

Let $S$ be a Dedekind scheme of positive dimension; that is, $S$ is a normal noetherian scheme of dimension 1. (We will be concerned in this chapter with the local behavior of the Flach map, so there is no need to consider the case where $S$ is the spectrum of a field.) We assume further that $S$ is connected; since it is normal this implies that it is irreducible. Let $F$ be the function field (i.e., the residue field at the generic point) of $S$.

Let $\mathfrak{X}$ be a smooth proper $S$-scheme of relative dimension $n$. Write $X$ for the generic fibre $\mathfrak{X}\times_S \operatorname{Spec} F$ of $\mathfrak{X}$. Fix an integer $N$ relatively prime to the characteristic of $F$ and let $\mathcal{F}$ be the sheaf $\mathbf{Z}/N\mathbf{Z}$ on $\mathfrak{X}$. Fix another integer $m$, $0 \leq m \leq n$ and let $V$ denote $H^{2m}(X_{F_{\mathrm{s}}}, \mathcal{F}(m+1))$, considered as a $G_F$-module. If we assume that $X$ satisfies purity at $N$, then we have a Flach map

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}} \to H^1(F, V)$$

as in Section VI.5.

Fix now a closed point $v$ of $S$ with residue characteristic prime to $N$; $\mathcal{O}_{S,v}$ is a discrete valuation ring. Let $R$ denote its completion, with residue field $k$ and fraction field $K$. In this chapter we will give a description of the local behavior of the cohomology classes coming from $\sigma_m$ at the closed point $v$. Note that by smooth base change and the local constancy of higher direct images [**FK88**, Chapter I, Theorem 8.9], $V$ is unramified as a $G_K$-module and there is an isomorphism

$$V \cong H^{2m}\big(\mathfrak{X}_{k_{\mathrm{s}}}, \mathcal{F}(m+1)\big).$$

For $(Z, f) \in E_2^{m,-m-1}(X)$, let $\bar{Z}$ denote the closure of $Z$ in $\mathfrak{X}_R$. Let

$$\operatorname{div}_k : E_2^{m,-m-1}(X) \to A^m\mathfrak{X}_k$$

be the map sending $(Z, f)$ to the divisor of $f$ on $\bar{Z}$; this divisor is supported entirely on $\bar{Z}_k$ (see (VI.5.3)) and has codimension $m$ in $\mathfrak{X}_k$. Let $H^1_s(K, V)$ denote the group $H^0(k, H^1(\mathcal{I}_K, V))$ where $\mathcal{I}_K$ is the inertia group of $K$. Recall that there is a canonical isomorphism

$$V(-1)^{G_k} \cong H^1_s(K, V)$$

as in Lemma I.1.3. The goal of this chapter is to give a proof of the following theorem.

THEOREM 1.1. *Let $S$, $\mathfrak{X}$, $\mathcal{F}$ and $v$ be as above. Assume also that $X_K$ and $\mathfrak{X}_k$ satisfy purity at $N$, and that the pair $\mathfrak{X}_R, \mathfrak{X}_k$ satisfies relative purity at $N$. Then there is a commutative diagram*

$$\begin{array}{ccc}
E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{\mathrm{div}_k} & A^m\mathfrak{X}_k \\
\Big\downarrow{\scriptstyle \sigma_m} & & \Big\downarrow{\scriptstyle s} \\
H^1(F,V) & & H^{2m}\big(\mathfrak{X}_{k_s}, \mathcal{F}(m)\big)^{G_k} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\
H^1(K,V) & \longrightarrow & H^1_s(K,V)
\end{array}$$

*Here $s$ is the cycle class map in étale cohomology, the unlabelled maps are the canonical restriction and singular restriction maps, and the unlabelled isomorphism is the isomorphism of Lemma I.1.3. The same diagram commutes if $\mathcal{F}$ is a constant sheaf of $\mathbf{Z}_l$-modules or $\mathbf{Q}_l$-vector spaces.*

## 2. Local behavior I

Let $\mathcal{F}$ be a fixed constant $N$-torsion sheaf on $\mathfrak{X}$. The first step in the proof of Theorem 1.1 is to connect the Flach map over $F$ with the local Flach map over $K$. This is easy; indeed, by our assumption that $X_K$ satisfies purity at $N$, the Flach map

$$E_2^{m,-m-1}(X_K)_{0,\mathcal{F}} \to H^1\big(\operatorname{Spec} K, R^{2m}u'_{v*}\mathcal{F}(m+1)\big), \tag{2.1}$$

is defined over $X_K$; here $u'_v : X_K \to \operatorname{Spec} K$ is the structure map.

We now relate (2.1) with the global Flach map. Specifically, we have the commutative diagram (VI.7.1):

$$\begin{array}{ccc}
E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{g_1} & E_2^{m,-m-1}(X_K)_{0,\mathcal{F}} \\
\Big\downarrow{\scriptstyle c(F)} & & \Big\downarrow{\scriptstyle c(K)} \\
E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{g_2} & E_2^{m,m+1}\big(X_K, \mathcal{F}(m+1)\big)_0 \\
\Big\downarrow{\scriptstyle d(F)} & & \Big\downarrow{\scriptstyle d(K)} \\
H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{g_3} & H^{2m+1}\big(X_K, \mathcal{F}(m+1)\big)_0 \\
\Big\downarrow{\scriptstyle e(F)} & & \Big\downarrow{\scriptstyle e(K)} \\
H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big) & \xrightarrow{g_4} & H^1\big(\operatorname{Spec} K, R^{2m}u'_{v*}\mathcal{F}(m+1)\big)
\end{array} \tag{2.2}$$

Here we have labelled the maps in the definition of the Flach map by the base scheme, rather than the scheme itself; we will continue to do so for the remainder of this chapter.

## 3. Local behavior II

The next step is to connect the local Flach map to a relative Flach-type map for the pair $\mathfrak{X}_R$, $\mathfrak{X}_k$. Here we need the assumption that this pair satisfies relative purity at $N$. Note that $\mathfrak{X}_R$ is at least regular over $R$, since $\mathfrak{X}$ is smooth over $S$.

At this point it is crucial that we are working with the $E_2$-terms of the coniveau spectral sequence. We will construct a commutative diagram

$$
\begin{array}{ccc}
E_2^{m,-m-1}(X_K)_{0,\mathcal{F}} & \xrightarrow{\quad\delta_1\quad} & E_{2,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}_R) \\
\Big\downarrow{\scriptstyle c(K)} & & \Big\downarrow{\scriptstyle c(R,k)} \\
E_2^{m,m+1}\big(X_K,\mathcal{F}(m+1)\big)_0 & \xrightarrow{\quad\delta_2\quad} & E_{2,\mathfrak{X}_k}^{m+1,m+1}\big(\mathfrak{X}_R,\mathcal{F}(m+1)\big) \\
\Big\downarrow{\scriptstyle d(K)} & & \Big\downarrow{\scriptstyle d(R,k)} \\
H^{2m+1}\big(X_K,\mathcal{F}(m+1)\big)_0 & \xrightarrow{\quad\delta_3\quad} & H_{\mathfrak{X}_k}^{2m+2}\big(\mathfrak{X}_R,\mathcal{F}(m+1)\big) \\
\Big\downarrow{\scriptstyle e(K)} & & \Big\downarrow{\scriptstyle e(R,k)} \\
H^1\big(\operatorname{Spec} K, R^{2m}u'_{v*}\mathcal{F}(m+1)\big) & \xrightarrow{\ \delta_4\ } & H^2_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m}u_{v*}\mathcal{F}(m+1)\big)
\end{array}
\tag{3.1}
$$

where $u_v : \mathfrak{X}_R \to \operatorname{Spec} R$ is the structure map.

The left-hand vertical maps have already been defined. $\delta_1$ and $\delta_2$ are the localization maps of Lemma VI.2.1 and Proposition VI.4.1 for $\mathfrak{X}_k \hookrightarrow \mathfrak{X}_R$; note that $\mathfrak{X}_K$ is dense in $\mathfrak{X}_R$, so that these maps do exist. $c(R,k)$ is Gillet's Chern class map, and this square commutes by Proposition VI.4.2.

$d(R,k)$ is defined as an edge map in the same way as $d(F)$ and $d(K)$, but its existence requires our purity hypothesis on $\mathfrak{X}_R$. Given this assumption, the definition is almost the same as in the previous two cases: we can write

$$E_{1,\mathfrak{X}_k}^{pq}\big(\mathfrak{X}_R,\mathcal{F}(m+1)\big) = \bigoplus_{x\in\mathfrak{X}_R^p\cap\mathfrak{X}_k} H^{q-p}\big(\operatorname{Spec} k(x),\mathcal{F}(m+1-p)\big)$$

and this clearly vanishes for $q<p$. It follows from Example A.2.1 that the desired edge map $d(R,k)$ exists. $\delta_3$ is the localization map in étale cohomology coming from the long exact sequence of the pair $\mathfrak{X}_k \hookrightarrow \mathfrak{X}_R$. This square commutes by Lemma VI.2.1.

$e(R,k)$ is an edge map in the Leray spectral sequence with supports:

$$H^p_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^q u_{v*}\mathcal{F}(m+1)\big) \Rightarrow H^{p+q}_{\operatorname{Spec}\mathfrak{X}_k}\big(\operatorname{Spec}\mathfrak{X}_R,\mathcal{F}(m+1)\big).$$

To show that it exists we must show that

$$H^0_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m+2}u_{v*}\mathcal{F}(m+1)\big) = H^1_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m+1}u_{v*}\mathcal{F}(m+1)\big) = 0.$$

But this is immediate from purity as in Theorem VI.3.2(c) with $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} k$ (together with the fact that the higher direct images of the proper map $u_{v*}$ are locally constant). Indeed, purity identifies these with cohomology groups over $\operatorname{Spec} k$ in dimensions $-2$ and $-1$, which automatically vanish. $\delta_4$ is again a localization map in étale cohomology, together with a base change map.

That this square commutes follows from Proposition A.7.1. Specifically, take $G = u_{v*}$, $F_1 = \Gamma(\operatorname{Spec} k, i^!\cdot)$, $F_2 = \Gamma(\operatorname{Spec} R,\cdot)$, $F_3 = \Gamma(\operatorname{Spec} K,\cdot)$, where $i : \operatorname{Spec} k \to \operatorname{Spec} R$ is the natural map. The sequence

$$0 \to F_1(\mathcal{G}) \to F_2(\mathcal{G}) \to F_3(\mathcal{G}) \to 0$$

is left exact for any sheaf $\mathcal{G}$ and since restriction maps are surjective for injective sheaves the sequence is exact on injectives. Proposition A.7.1 now applies; the fact that the maps obtained agree with the usual boundary maps is a standard exercise in injective resolutions and is left to the reader.

## 4. Local behavior III

We now use purity to connect the relative Flach map of the previous section to a "lower" Flach map over the residue field $k$. We now need our assumption that $\mathfrak{X}_k$ satisfies purity at $N$. With this assumption, there is a commutative diagram

$$\begin{array}{ccc}
E_{2,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}_R) & \xleftarrow[\simeq]{p_1} & E_2^{m,-m}(\mathfrak{X}_k) \\
\downarrow{\scriptstyle c(R,k)} & & \downarrow{\scriptstyle c(k)} \\
E_{2,\mathfrak{X}_k}^{m+1,m+1}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big) & \xleftarrow[\simeq]{p_2} & E_2^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \\
\downarrow{\scriptstyle d(R,k)} & & \downarrow{\scriptstyle d(k)} \\
H_{\mathfrak{X}_k}^{2m+2}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big) & \xleftarrow[\simeq]{p_3} & H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \\
\downarrow{\scriptstyle e(R,k)} & & \downarrow{\scriptstyle e(k)} \\
H_{\operatorname{Spec} k}^2\big(\operatorname{Spec} R, R^{2m}u_{v*}\mathcal{F}(m+1)\big) & \xleftarrow[\simeq]{p_4} & H^0\big(\operatorname{Spec} k, R^{2m}u''_{v*}\mathcal{F}(m)\big)
\end{array} \tag{4.1}$$

where $u''_v : \mathfrak{X}_k \to \operatorname{Spec} k$ is the structure map and all horizontal maps are isomorphisms.

Again, the left-hand vertical maps have all already been defined. The right-hand vertical maps are not difficult to define: $c(k)$ is Gillet's Chern class map for $\mathfrak{X}_k$, $d(k)$ is defined as an edge map in the usual way (using purity for $\mathfrak{X}_k$) and $e(k)$ is an edge map in the Leray spectral sequence for $u''_v$.

The horizontal maps are all incarnations of purity. Recall that the spectral sequence $E_r^{pq}(\mathfrak{X}_k)$ is constructed from the filtration of coherent sheaves on $\mathfrak{X}_k$ by codimension of support; see [**Qui73**, Section 7, Theorem 5.4]. Similarly, $E_{r,\mathfrak{X}_k}^{pq}(\mathfrak{X}_R)$ is constructed from the filtration of coherent sheaves on $\mathfrak{X}_R$, supported on $\mathfrak{X}_k$, by codimension of support. With these descriptions it is clear that these spectral sequences coincide, with a shift in the indices; that is, there is an isomorphism of spectral sequences

$$E_r^{pq}(\mathfrak{X}_k) \to E_{r,\mathfrak{X}_k}^{p+1,q-1}(\mathfrak{X}_R). \tag{4.2}$$

At the first stage, this coincides with the obvious identification of $E_1$-terms:

$$E_1^{pq}(\mathfrak{X}_k) = \bigoplus_{x \in \mathfrak{X}_k^p} K_{-p-q}k(x) = \bigoplus_{x \in \mathfrak{X}_R^{p+1} \cap \mathfrak{X}_k} K_{-p-q}k(x) = E_{1,\mathfrak{X}_k}^{p+1,q-1}(\mathfrak{X}_R).$$

We let $p_1$ be the bidegree $(m,-m)$ isomorphism in (4.2) with $r = 2$; at the first stage this looks like

$$\begin{array}{ccc}
E_1^{m,-m}(\mathfrak{X}_k) & \longrightarrow & E_{1,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}_R) \\
\| & & \| \\
\bigoplus_{x \in \mathfrak{X}_k^m} K_0 k(x) & = & \bigoplus_{x \in \mathfrak{X}_R^{m+1} \cap \mathfrak{X}_k} K_0 k(x)
\end{array}$$

Next, recall that by relative purity for $\mathfrak{X}_R$, $\mathfrak{X}_k$ we have isomorphisms

$$H_Z^i\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \to H_Z^{i+2}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big)$$

for all irreducible regular closed subschemes $Z$ of $\mathfrak{X}_k$. These compile to isomorphisms

$$H^i\big(\mathfrak{X}_k, \mathcal{F}(m)\big)^p \to H^{i+2}_{\mathfrak{X}_k}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big)^{p+1}$$

in the notation of Section VI.1. These identifications yield isomorphisms of the exact couples (VI.1.3) used to define the coniveau spectral sequence, with a shift in indices, and thus an isomorphism

$$E_r^{pq}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \to E_{r,\mathfrak{X}_k}^{p+1,q+1}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big). \tag{4.3}$$

At the first stage, this coincides with the previous purity identification

$$E_1^{pq}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) = \bigoplus_{x\in\mathfrak{X}_k^p} H^0\big(k(x), \mathcal{F}(m-p)\big) = \bigoplus_{x\in\mathfrak{X}_R^{p+1}\cap\mathfrak{X}_k} H^0\big(k(x), \mathcal{F}(m-p)\big) = E_{1,\mathfrak{X}_k}^{p+1,q+1}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big).$$

We let $p_2$ be the bidegree $(m, m)$ isomorphism of (4.3) with $r = 2$; at the first stage it is just

$$\begin{array}{ccc} E_1^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) & \longrightarrow & E_{1,\mathfrak{X}_k}^{m+1,m+1}\big(\mathfrak{X}_R, \mathcal{F}(m+1)\big) \\ \| & & \| \\ \bigoplus_{x\in\mathfrak{X}_k^m} H^0(k(x), \mathcal{F}) & = & \bigoplus_{x\in\mathfrak{X}_R^{m+1}\cap\mathfrak{X}_k} H^0(k(x), \mathcal{F}) \end{array}$$

That the first square in (4.1) commutes is now immediate from the description of the Chern class maps given in (VI.4.5). Indeed, in both case these maps are induced by divisor maps which themselves coincide.

$p_3$ is just the usual purity isomorphism; the second square commutes by Proposition A.8.1 and the fact that $p_2$ is defined in terms of purity.

To define $p_4$ we first must define an isomorphism of spectral sequences

$$H^p\big(\operatorname{Spec} k, R^q u''_{v*}\mathcal{F}(m)\big) \to H^{p+2}_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^q u_{v*}\mathcal{F}(m+1)\big). \tag{4.4}$$

This is most easily defined using the construction of the Leray spectral sequence and the description of purity given in [**Gil81**, pp. 205-207, esp. p. 207(vi)]. $p_4$ is the second stage bidegree $(0, 2m)$ part of (4.4). The desired commutativity is just the standard compatibility of Leray spectral sequences and edge maps, taking into account the shift in indices.

## 5. The divisor map

The diagrams (2.2), (3.1) and (4.1) combine to put the Flach map in a commutative diagram

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \longrightarrow & E_2^{m,-m}(\mathfrak{X}_k) \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow \\ H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big) & \longrightarrow & H^0\big(\operatorname{Spec} k, R^{2m}u''_{v*}\mathcal{F}(m)\big) \end{array} \tag{5.1}$$

We now need to evaluate the other three maps.

We begin with the top horizontal map

$$E_2^{m,-m-1}(X)_{0,\mathcal{F}} \xrightarrow{g_1} E_2^{m,-m-1}(X_K)_{0,\mathcal{F}} \xrightarrow{\delta_1} E_{2,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}_R) \xleftarrow{p_1} E_2^{m,-m}(\mathfrak{X}_k). \tag{5.2}$$

Evaluating the four terms (using Proposition VI.4.1 and the computation of the $K$-groups of a field) identifies (5.2) with a subquotient of a diagram

$$\bigoplus_{x\in X^m} k(x)^\times \to \bigoplus_{x\in X_K^m} k(x)^\times \dashrightarrow \bigoplus_{x\in \mathfrak{X}_R^{m+1}\cap\mathfrak{X}_k} \mathbf{Z} = \bigoplus_{x\in\mathfrak{X}_k^m} \mathbf{Z}. \tag{5.3}$$

Here the first and last maps exist at the first stage, but the middle one does not exist until the second stage.

Consider a codimension $m$ cycle $x$ on $X$ and a rational function $f$ on $\bar{x}$. The $K$-scheme $\bar{x} \times_{\operatorname{Spec} F} \operatorname{Spec} K$ has a finite number of irreducible components $\bar{x}_i$ of codimension $m$ [**GDb**, Corollary 4.5.10], and there are natural inclusions $k(x) \hookrightarrow k(x_i)$ of function fields. Under $g_1$ in (5.3), the pair $(x, f)$ maps to $\sum(x_i, f)$, where by $f$ we mean the rational function on $x_i$ coming from the inclusion of function fields above. This description follows from the fact that these maps agree with the natural maps $K_1k(x) \to \oplus K_1k(x_i)$, which are as we just described.

$\delta_1$ is a boundary map in the exact localization sequence. Recall that this boundary map is computed from the diagram

$$\begin{array}{ccccc}
E_{1,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}_R) & \longrightarrow & E_1^{m+1,-m-1}(\mathfrak{X}_R) & \longrightarrow & E_1^{m+1,-m-1}(X_K) \\
\uparrow & & \uparrow & & \uparrow \\
E_{1,\mathfrak{X}_k}^{m,-m-1}(\mathfrak{X}_R) & \longrightarrow & E_1^{m,-m-1}(\mathfrak{X}_R) & \longrightarrow & E_1^{m,-m-1}(X_K)
\end{array}$$

by pulling back, pushing forward and pulling back. Pulling back a pair $(x_i, f)$ on $X_K$ to $\mathfrak{X}_R$ maps it to the pair $(\bar{x}_i, f)$ on $\mathfrak{X}_R$, where $\bar{x}_i$ is the closure in $\mathfrak{X}_R$ of $x_i$ and $f$ is regarded as a rational function on $\bar{x}_i$. By Proposition VI.4.1, the differential at this stage is identified with the divisor map, so pushing this forward yields the divisor $\operatorname{div}_{\mathfrak{X}_R}(f)$ of the rational function $f$ on $\mathfrak{X}_R$. Since we are assuming that $(x, f)$ lies in $E_2^{m,-m-1}(X_K)$ and thus that the divisor of $f$ has no intersection with $X_K$ (see (VI.5.3)), this divisor is supported on $\mathfrak{X}_k$ and thus pulls back. In summary, the image of $(x_i, f)$ under $\delta_1$ is nothing more than the divisor of $f$ on $\mathfrak{X}_R$, which is necessarily supported on $\mathfrak{X}_k$.

$p_1$ is just the canonical identification of terms given in Section 4; this reinterprets the divisor above back on $\mathfrak{X}_k$. It follows, then, that the map

$$E_2^{m,-m-1}(X) \to E_2^{m,-m}(\mathfrak{X}_k)$$

of (5.2) sends a pair of a codimension $m$ cycle $x$ and a rational function $f \in k(x)^\times$ to the part of the divisor of $f$ which is supported on $\mathfrak{X}_k$. This is the map which we previously denoted $\operatorname{div}_k$.

## 6. The cycle map

We next compute the vertical map

$$E_2^{m,-m}(\mathfrak{X}_k) \xrightarrow{c(k)} E_2^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \xrightarrow{d(k)} H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \xrightarrow{e(k)} H^0\big(\operatorname{Spec} k, R^{2m}u''_{v*}\mathcal{F}(m)\big).$$

Recall that we can evaluate the first two terms as subquotients of

$$\bigoplus_{x\in\mathfrak{X}_k^m} K_0 k(x) \to \bigoplus_{x\in\mathfrak{X}_k^m} H^0(k(x), \mathcal{F}), \tag{6.1}$$

and the map $K_0k(x) = \mathbf{Z} \to M = H^0(k(x), \mathcal{F})$ (where $M$ is the $\mathbf{Z}/N\mathbf{Z}$-module to which the constant sheaf $\mathcal{F}$ is associated) is just the natural map; see Proposition VI.4.2. That is, (6.1) sends an element $(x, 1)$ of $E_2^{m,-m}(\mathfrak{X}_k)$ to the "same" element $(x, 1)$ of $E_2^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big)$.

We next consider the edge map

$$E_2^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \xrightarrow{d(k)} H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \tag{6.2}$$

which we consider as the map induced from a first stage map

$$\bigoplus_{x\in\mathfrak{X}_k^m} H^0(k(x), \mathcal{F}) \to \bigoplus_{x\in\mathfrak{X}_k^m} H_x^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \to H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big). \tag{6.3}$$

Recall that

$$H_x^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) = \varinjlim_{Z\subsetneq\bar{x}} H_{\bar{x}-Z}^{2m}\big(\mathfrak{X}_k - Z, \mathcal{F}(m)\big). \tag{6.4}$$

For each $Z$ we have a well-defined fundamental class of $\bar{x} - Z$ in $H_{\bar{x}-Z}^{2m}\big(\mathfrak{X}_k - Z, \mathcal{F}(m)\big)$, and these classes are compatible with the maps of the direct system; see [**FK88**, Chapter II, Corollary 2.3]. They therefore compile to give an element $\xi_x$ of (6.4). Since for sufficiently large $Z$ (specifically, large enough so that $\bar{x} - Z$ is regular) the purity map sends the element 1 of $H^0(k(x), \mathcal{F})$ to the fundamental class of $\bar{x} - Z$ in $H_{\bar{x}-Z}^{2m}(\mathfrak{X}_k - Z, \mathcal{F}(m))$, we see that under the first map of (6.3) the element $(x, 1)$ must map to $\xi_x \in H_x^{2m}(\mathfrak{X}_k, \mathcal{F}(m))$. But $\xi_x$ can already be realized as the fundamental class of $\bar{x}$ at the initial $Z = \emptyset$ term $H_{\bar{x}}^{2m}(\mathfrak{X}_k, \mathcal{F}(m))$ of the direct limit, and the map

$$H_{\bar{x}}^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \to H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big)$$

of (6.3) is just the natural map. We conclude that $(x, 1) \in E_1^{m,-m}(\mathfrak{X}_k, \mathcal{F}(m))$ maps to the fundamental class of $\bar{x}$ in $H^{2m}(\mathfrak{X}_k, \mathcal{F}(m))$; the map (6.2) at the $E_2$-level has the same description.

For $e(k)$, we first identify $R^{2m}u''_{v*}\mathcal{F}(m)$ with $H^{2m}(\mathfrak{X}_{k_s}, \mathcal{F}(m))$. Under this identification, we are considering a map

$$H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \to H^{2m}\big(\mathfrak{X}_{k_s}, \mathcal{F}(m)\big)^{G_k} \tag{6.5}$$

which is nothing more than the map induced from the map $\mathfrak{X}_{k_s} \to \mathfrak{X}_k$; see [**Wei94**, Section 5.8]. By transitivity of the fundamental class this sends the fundamental class of $\bar{x}$ in $\mathfrak{X}_k$ to the fundamental class of $\bar{x} \times_{\operatorname{Spec} k} \operatorname{Spec} k_s$ in $\mathfrak{X}_{k_s}$; it is necessarily Galois invariant since it comes from a cycle defined over $k$.

Combining our descriptions of (6.1), (6.2) and (6.5), we see that the map

$$\bigoplus_{x\in\mathfrak{X}_k^m} \mathbf{Z} \twoheadrightarrow E_2^{m,-m}(\mathfrak{X}_k) \to H^{2m}(\mathfrak{X}_{k_s}, \mathcal{F}(m))^{G_k}$$

sends an element with a 1 corresponding to $x \in \mathfrak{X}_k^m$ and zero everywhere else to the fundamental class of $\bar{x} \times_{\operatorname{Spec} k} \operatorname{Spec} k_{\mathrm{s}}$; the general definition follows by linearity. Identifying $E_2^{m,-m}(\mathfrak{X}_k)$ with the Chow group $A^m\mathfrak{X}_k$ as in Proposition VI.4.1, we will write this cycle map as

$$s : A^m(\mathfrak{X}_k) \to H^{2m}\big(\mathfrak{X}_{k_{\mathrm{s}}}, \mathcal{F}(m)\big)^{G_k}.$$

## 7. Relations with Galois cohomology

The last map in (5.1) to identify is the bottom map

$$H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big) \xrightarrow{g_4} H^1\big(\operatorname{Spec} K, R^{2m}u'_{v*}\mathcal{F}(m+1)\big) \xrightarrow{\delta_4}$$
$$H^2_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m}u_{v*}\mathcal{F}(m+1)\big) \xrightarrow{p_4} H^0\big(\operatorname{Spec} k, R^{2m}u''_{v*}\mathcal{F}(m)\big).$$

We first evaluate some of the terms a bit more. $R^{2m}u_*\mathcal{F}(m+1)$ as an étale sheaf on $\operatorname{Spec} F$ corresponds to the $G_F$-module $H^{2m}(X_{F_{\mathrm{s}}}, \mathcal{F}(m+1))$. Similarly, the étale sheaf $R^{2m}u'_{v*}\mathcal{F}(m+1)$ on $\operatorname{Spec} K$ corresponds to $H^{2m}(X_{K_{\mathrm{s}}}, \mathcal{F}(m+1))$, which is a $G_K$-module. The smooth base change theorem [**FK88**, Chapter I, Section 8] shows that there is a natural isomorphism

$$H^{2m}\big(X_{F_{\mathrm{s}}}, \mathcal{F}(m+1)\big) \cong H^{2m}\big(X_{K_{\mathrm{s}}}, \mathcal{F}(m+1)\big)$$

as $G_K$-modules. Note that these are both the Galois module previously denoted $V$. Under these identifications, the base change map $g_4$ identifies with the usual restriction map $H^1(F, V) \to H^1(K, V)$ in Galois cohomology.

It remains to identify the sequence of maps

(7.1) $$H^1\big(\operatorname{Spec} K, R^{2m}u'_{v*}\mathcal{F}(m+1)\big) \xrightarrow{\delta_4} H^2_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m}u_{v*}\mathcal{F}(m+1)\big) \xrightarrow{p_4} H^0\big(\operatorname{Spec} k, R^{2m}u''_{v*}\mathcal{F}(m)\big).$$

The smooth base change theorem and the local constancy of higher direct images under proper maps ([**FK88**, Chapter I, Theorem 8.9]; it is here that it is crucial that $\mathfrak{X}$ is proper over $S$) shows that $V$ is unramified as a $G_K$-module, and identifies $R^{2m}u''_{v*}\mathcal{F}(m)$ with $V(-1)$ as a $G_k$-module. Making these identifications we can rewrite (7.1) as

(7.2) $$H^1(K, V) \to H^2_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m}u_{v*}\mathcal{F}(m+1)\big) \to H^0\big(k, V(-1)\big).$$

As in Section 1, the last term in (7.2) identifies with $H^1_s(K, V)$. Taking this into account, we see that we are trying to identify a map

(7.3) $$H^1(K, V) \to H^1_s(K, V).$$

Fortunately, Grothendieck showed that under these identifications, the map (7.3) which we are trying to evaluate really is nothing more than the natural singular restriction map; see [**CT95**, Section 3.3] and [**Gro77**, Exposé 1, pp. 50-52]. This completes the proof of Theorem 1.1 for torsion sheaves.

## 8. Functoriality and passage to the limit

We continue with our running hypotheses. Let $\mathcal{F}'$ be a second constant $N$-torsion sheaf on $\mathfrak{X}$ and suppose that we are given a map $\pi : \mathcal{F} \to \mathcal{F}'$. Set $V' =$

$H^{2m}(X_{F_s}, \mathcal{F}'(m+1))$. We now have two Flach maps fitting into commutative diagrams

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{\operatorname{div}_k} & A^m\mathfrak{X}_k \\ \sigma_m \big\downarrow & & \big\downarrow s \\ H^1(F,V) & & H^{2m}\big(\mathfrak{X}_{k_s}, \mathcal{F}(m)\big)^{G_k} \\ \big\downarrow & & \big\downarrow \simeq \\ H^1(K,V) & \longrightarrow & H^1_s(K,V) \end{array}$$

and

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}'} & \xrightarrow{\operatorname{div}_k} & A^m\mathfrak{X}_k \\ \sigma_m \big\downarrow & & \big\downarrow s \\ H^1(F,V') & & H^{2m}\big(\mathfrak{X}_{k_s}, \mathcal{F}'(m)\big)^{G_k} \\ \big\downarrow & & \big\downarrow \simeq \\ H^1(K,V') & \longrightarrow & H^1_s(K,V') \end{array}$$

These diagrams are connected by various obvious maps coming from $\pi$ and we claim that the resulting three-dimensional diagram is commutative.

This is not difficult; we checked that the Flach map is compatible with such maps in Section VI.6, and the rest of the commutativities are clear. Indeed, the only non-obvious one is the cycle map, which is proved in [**FK88**, Chapter II, Corollary 2.3]. Note that the somewhat daunting map $\operatorname{div}_k$ doesn't depend on $\mathcal{F}$ or $\mathcal{F}'$ at all.

In particular, assuming that the purity hypotheses are all satisfied and proceeding as in Section VI.6 we find that Theorem 1.1 holds for $l$-adic sheaves as well. Tensoring with $\mathbf{Q}_l$ yields the result for sheaves of $\mathbf{Q}_l$-vector spaces, and completes the proof of the theorem.

## 9. Example : Schemes over global fields

Let $F$ be a global field and let $\mathcal{O}_F$ denote its ring of integers. Let $S$ be an open subscheme of $\operatorname{Spec}\mathcal{O}_F$. In this section we restate Theorem 1.1 for schemes over $S$.

Let $\mathfrak{X}$ be a smooth projective scheme of relative dimension $n$ over $S$. Let $X$ be the generic fiber of $\mathfrak{X}$. Let $m$ be an integer $0 \le m \le n$, and let $l$ be a prime. If $F$ is of characteristic 0 we allow $l$ to be arbitrary, while if $F$ has positive characteristic we require $l$ to be relatively prime to the characteristic and greater than or equal to $n+2$. Set $V = H^{2m}(X_{\bar{F}}, \mathbf{Z}_l(m+1))$.

Theorem 9.1. *Let $F$, $S$, $\mathfrak{X}$ and $l$ be as above. Let $v$ be a place of $F$ lying in $S$ and of residue characteristic prime to $l$. Let $K$ denote the completion of $F$ at $v$*

*and let $k$ denote the residue field of $v$. Then there is a commutative diagram*

$$\begin{array}{ccc} E_2^{m,-m-1}(X)\otimes \mathbf{Q}_l & \xrightarrow{\mathrm{div}_k} & A^m\mathfrak{X}_k\otimes \mathbf{Q}_l \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle s} \\ H^1(F,V\otimes \mathbf{Q}_l) & & H^{2m}\big(\mathfrak{X}_{k_s},\mathbf{Q}_l(m)\big)^{G_k} \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ H^1(K,V\otimes \mathbf{Q}_l) & \longrightarrow & H^1_s(K,V\otimes \mathbf{Q}_l) \end{array}$$

*If $H^{2m+1}(X_{F_s},\mathbf{Z}_l)$ is torsion-free, then there is also a commutative diagram*

$$\begin{array}{ccc} E_2^{m,-m-1}(X) & \xrightarrow{\mathrm{div}_k} & A^m\mathfrak{X}_k \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle s} \\ H^1(F,V) & & H^{2m}\big(\mathfrak{X}_{k_s},\mathbf{Z}_l(m)\big)^{G_k} \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ H^1(K,V) & \longrightarrow & H^1_s(K,V) \end{array}$$

PROOF. Theorem VI.3.2 shows that all of the purity hypotheses are satisfied, so that the Flach maps exist. Theorem 1.1 now gives the diagrams above, with Lemma VI.6.2 identifying the 0-parts of the $E_2$-terms. □

## 10. Local behavior at places over $l$

In this section we give a result which will allow us to control the behavior of Flach classes at places dividing $l$, at least in certain (somewhat restrictive) circumstances. It would be useful to have a stronger result. Note that despite the local nature of the statement of the proposition, the proof will require the global hypothesis.

Let $S$ be an open subscheme of the spectrum of the ring of integers of a number field $F$ and let $\mathfrak{X}$ be a smooth proper scheme of relative dimension $n$ over $S$. Let $l$ be a prime and let $m$ be another integer $0 \le m \le n$. Let $X$ be the generic fiber of $\mathfrak{X}$.

Let $T$ be a quotient of $H^{2m}(X_{\bar{F}},\mathbf{Z}_l(m+1))$ as a $G_F$-module. Note that we can consider images of cycle classes of codimension $m$ in $T(-1)$. Fix a place $v$ dividing $l$ (although the statement below will not depend on which such $v$ we pick) and let $K$ be the completion of $F$ at $v$, with valuation ring $R$. We let $\sigma : E_2^{m,-m-1}(X) \to H^1(K,T)$ denote the composition of the Flach map $\sigma_m$ with projection to $T$ and restriction to $K$.

PROPOSITION 10.1. *Let $(Z,f)$ be an element of $E_2^{m,-m-1}(X)$ and let $\bar{Z}$ denote the closure of $Z_K$ in $\mathfrak{X}_R$. If the fundamental class of $Z_{K_s}$ in $X_{K_s}$ maps to $0$ in $T(-1)$, then $\sigma_m(Z,f)_v$ lies in $H^1_f(K,T)$ for the deRham local finite/singular structure at $v$. If further it is possible to realize $U = \mathfrak{X}_R - \bar{Z}$ as the complement of a normal crossings divisor in a smooth proper variety over $R$, then $\sigma(Z,f)$ lies in $H^1_f(K,T)$ for the crystalline finite/singular structure at $v$.*

For definitions of the local conditions in the proposition, see Section I.4. In particular, the definitions imply that it is enough to prove the results after tensoring by $\mathbf{Q}_l$.

Proof. For ease of notation, set

$$\begin{aligned} H^i_Z &= H^i_{Z_K}\big(X_K, \mathbf{Q}_l(m+1)\big) \\ H^i &= H^i\big(X_K, \mathbf{Q}_l(m+1)\big) \\ \bar{H}^i_Z &= H^i_{Z_{K_s}}\big(X_{K_s}, \mathbf{Q}_l(m+1)\big) \\ \bar{H}^i &= H^i\big(X_{K_s}, \mathbf{Q}_l(m+1)\big). \end{aligned}$$

We have the following commutative diagram of Flach and Jannsen (see [**Jan90**, Part II, Lemma 9.5] and [**Fla92**, pp. 323–324]):

$$\begin{array}{ccc} \ker\big(H^{2m+1}_Z \to (H^{2m+1})^{G_K}\big) & \longrightarrow & \ker\big(H^{2m+1} \to (\bar{H}^{2m+1})^{G_K}\big) \\ \downarrow & & \downarrow{\scriptstyle e(X_K)} \\ \ker\big(\bar{H}^{2m+1}_Z \to \bar{H}^{2m+1}\big)^{G_K} & & H^1(K, \bar{H}^{2m}) \\ & \searrow^{\delta} & \downarrow \\ & & H^1(K, \bar{H}^{2m}/\bar{H}^{2m}_Z) \\ & & \downarrow \\ & & H^1(K, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l) \end{array}$$

where $e(X_K)$ is the map of Section VI.5; $\delta$ is a boundary map in the Galois cohomology sequence of

$$0 \to \bar{H}^{2m}/\bar{H}^{2m}_Z \to H^{2m}\big(U_{K_s}, \mathbf{Q}_l(m+1)\big) \to \ker\big(\bar{H}^{2m+1}_Z \to \bar{H}^{2m+1}\big) \to 0; \tag{10.1}$$

and the map to $H^1(K, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l)$ exists since $\bar{H}^{2m}_Z$ is generated by the (twisted) fundamental class of $Z_{K_s}$ in $X_{K_s}$, which vanishes in $T$ by hypothesis.

Let $d(X_K)$ and $c(X_K)$ denote the maps of Section V.5. The element

$$d(X_K) \circ c(X_K)(Z_K, f) \tag{10.2}$$

of $H^{2m+1}$ maps to 0 in $(\bar{H}^{2m+1})^{G_K}$ since it can be lifted to a global element which factors through $H^{2m+1}(X_{\bar{F}}, \mathbf{Q}_l(m+1))^{G_F} = 0$. It is also clear from the construction of the coniveau spectral sequence that (10.2) arises from an element of $H^{2m+1}_Z$. Let $\tau$ denote the image of this element of $H^{2m+1}_Z$ in $\ker(\bar{H}^{2m+1}_Z \to \bar{H}^{2m+1})^{G_K}$.

Let $\sigma'$ denote the image of $\sigma_m(Z,f)_v$ in $H^1(K, \bar{H}^{2m}/\bar{H}^{2m}_Z)$; it is just the image of (10.2). Using the Yoneda extension interpretation of $H^1(K, \bar{H}^{2m}/\bar{H}^{2m}_Z)$, the element $\sigma'$ corresponds to a $G_K$-module extension of $\mathbf{Q}_l$ by $\bar{H}^{2m}/\bar{H}^{2m}_Z$. But by the above argument, $\sigma'$ is equal to $\delta(\tau)$; the extension interpretation of $\delta$ thus implies that the extension corresponding to $\sigma'$ is obtained from (10.1) by pullback via the map

$$\mathbf{Q}_l\tau \hookrightarrow \ker(\bar{H}^{2m+1}_Z \to \bar{H}^{2m+1}).$$

The element $\sigma(Z,f) \otimes \mathbf{Q}_l \in H^1(K, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l)$ arises from this extension by pushout via $\bar{H}^{2m}/\bar{H}^{2m}_Z \to T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. We conclude from (10.1) that the extension corresponding to $\sigma(Z,f) \otimes \mathbf{Q}_l$ can be realized as a subquotient of $H^{2m}(U_{K_s}, \mathbf{Q}_l(m+1))$.

By [**Fal89**, Theorem 5.3 and Theorem 8.1], under the appropriate hypotheses $H^{2m}(U_{K_s}, \mathbf{Q}_l(m+1))$ is deRham or even crystalline; since these properties are preserved under passage to subquotients, the characterization of deRham and crystalline structures in terms of extensions [**BK90**, (3.7)] completes the proof. □

In fact, recent results of Kisin [**Kis**] show that we could replace the deRham structure above with the potentially semistable structure. Since our applications will only involve potentially semistable representations (for which the deRham and potentially semistable structures coincide), we will not make any further use of this result.

CHAPTER 8

# Flach classes for correspondences

In the first half of this chapter we study algebraic correspondences and the corresponding operations on algebraic $K$-theory and étale cohomology, culminating in the Leibniz relation of Theorem 6.1. We then apply this theory to set-up the methods for the production of cohesive Flach systems via the Flach map.

## 1. Algebraic correspondences

In this chapter we will describe the additional algebraic structure on Flach classes associated to algebras of self-correspondences on varieties. With a view towards our intended applications we will work in a fairly restricted setting. It seems likely that many of the results of this chapter remain true more generally, but I have not attempted a proper formulation.

Let $X$ and $Y$ be smooth proper varieties over a number field $F$. (For the first three sections we will only need that $F$ is perfect, but for the remainder of the chapter we will be using Lemma VI.6.2 in an essential way.) All products in this chapter will be over $\operatorname{Spec} F$ or $\operatorname{Spec} \bar{F}$ unless otherwise noted; it should be clear from context which is meant. We assume further that both $X$ and $Y$ have the same dimension $n$.

Definition 1.1. An *irreducible correspondence from $X$ to $Y$* is a subscheme $\alpha \hookrightarrow X \times Y$ such that the projections $\pi_{\alpha X} : \alpha \to X$ and $\pi_{\alpha Y} : \alpha \to Y$ are both finite and faithfully flat. A general *correspondence from $X$ to $Y$* is a formal sum (with integer coefficients, or later with $\mathbf{Z}_l$-coefficients) of such irreducible correspondences.

Note that an algebraic correspondence from $X$ to $Y$ necessarily has dimension $n$. Of course, the terminology "from $X$ to $Y$" is introduced purely for notational reasons. Note also that our definition is much less general than that in [**Ful98**, Chapter 16]. The difference lies in the fact that for us it will not be enough to work modulo rational equivalence.

If $\alpha \hookrightarrow X \times Y$ is an arbitrary closed subscheme such that the maps from each irreducible component of $\alpha$ to $X$ and $Y$ are finite and faithfully flat, we define the associated correspondence as follows: Let $\alpha_1, \dots, \alpha_r$ be the irreducible components of $\alpha$. For each $\alpha_i$, let $m_i$ be the length of the local ring at the generic point of $\alpha_i$; the associated correspondence, which we will also denote $\alpha$, is then $\sum m_i \alpha_i$.

We will use algebraic correspondences to define maps in $K$-theory and étale cohomology. Let $\alpha$ be an irreducible correspondence from $X$ to $Y$, with projections $\pi_{\alpha X}$ and $\pi_{\alpha Y}$. Given an étale sheaf $\mathcal{F}$ on $\alpha$, we define a map

$$\alpha_* : H^i(X, \pi_{\alpha X}^* \mathcal{F}) \to H^i(\alpha, \mathcal{F}) \to H^i(Y, \pi_{\alpha Y}^* \mathcal{F});$$

here the first map is the usual contravariant map on étale cohomology, and the second map is the trace map. Since $\pi_{\alpha X}$ and $\pi_{\alpha Y}$ respect codimensions one sees immediately that we obtain maps of the exact couples (VI.1.3) used to define the coniveau spectral sequence; this yields a map of spectral sequences which we also denote $\alpha_*$:

$$\alpha_* : E_r^{pq}(X, \pi_{\alpha X}^* \mathcal{F}) \to E_r^{pq}(\alpha, \mathcal{F}) \to E_r^{pq}(Y, \pi_{\alpha Y}^* \mathcal{F}).$$

Redoing the constructions in the opposite direction, we obtain maps

$$\alpha^* : H^i(Y, \pi_{\alpha Y}^* \mathcal{F}) \to H^i(X, \pi_{\alpha X}^* \mathcal{F})$$

$$\alpha^* : E_r^{pq}(Y, \pi_{\alpha Y}^* \mathcal{F}) \to E_r^{pq}(X, \pi_{\alpha X}^* \mathcal{F}).$$

We can also apply these constructions over $\bar{F}$ to obtain maps which we again denote

$$\alpha_* : H^i(X_{\bar{F}}, \pi_{X\alpha}^* \mathcal{F}_{\bar{F}}) \to H^i(Y_{\bar{F}}, \pi_{\alpha Y}^* \mathcal{F}_{\bar{F}})$$

$$\alpha^* : H^i(Y_{\bar{F}}, \pi_{Y\alpha}^* \mathcal{F}_{\bar{F}}) \to H^i(X_{\bar{F}}, \pi_{\alpha X}^* \mathcal{F}_{\bar{F}}).$$

Note that these last two maps commute with the action of $G_F$ since $\alpha$ is defined over $F$. They therefore can be used to induce maps on Galois cohomology. One checks immediately that these constructions are compatible with the natural map from étale cohomology over $F$ to étale cohomology over $\bar{F}$.

We obtain analogous maps

$$\alpha_* : E_r^{pq}(X) \to E_r^{pq}(Y)$$

$$\alpha^* : E_r^{pq}(Y) \to E_r^{pq}(X)$$

in $K$-theory using the appropriate contravariant and covariant functoriality. For an explicit description in the case which we will need, the map $\alpha_*$ on $E_1^{p,-p-1}$-terms

$$\bigoplus_{x \in X^p} k(x)^\times \to \bigoplus_{a \in \alpha^p} k(a)^\times \to \bigoplus_{y \in Y^p} k(y)^\times \tag{1.1}$$

is as follows: an element $(x, f)$ of the first direct sum maps to $\sum (a_i, \pi_{a_i X}^* f)$, where the sum runs over $a_i \in \pi_{\alpha X}^{-1}(x)$; note that each of these has codimension $p$ since $\pi_{\alpha X}$ is faithfully flat. The maps $\pi_{a_i X}^*$ are the natural inclusions $k(x) \hookrightarrow k(a_i)$. An element $(a, f)$ of the second direct sum in (1.1) maps to $(y, N_{k(a)/k(y)} f)$, where $y = \pi_{\alpha Y}(a)$ and $N_{k(a)/k(y)}$ is the norm mapping for the finite extension of fields $k(y) \hookrightarrow k(a)$. The map $\alpha^*$ on $E_1^{p,-p-1}$-terms has a similar description.

We extend the definitions above to general correspondences (with integer coefficients) by linearity. Note also that if the sheaf $\mathcal{F}$ is $l$-adic, then we can extend the operations on étale cohomology to correspondences with $\mathbf{Z}_l$-coefficients.

## 2. Correspondences and operations on étale cohomology

In this section we check that maps coming from correspondences are compatible with various maps in étale cohomology. We prove all results only for irreducible correspondences, but in each case they extend immediately to general correspondences by linearity. In order to discuss the theory with $\mathbf{Z}_l$-coefficients we will need the following definition.

Definition 2.1. Let $X$ be a variety over $F$. We say that $X$ is *cohomologically torsion-free at $l$* if the étale cohomology groups $H^i(X_{\bar{F}}, \mathbf{Z}_l)$ are torsion-free for all $i$.

Note that the Künneth theorem shows that if $X$ and $Y$ are cohomologically torsion-free at $l$, then so is $X \times Y$.

**2.1. Künneth projections.** Let $X$, $Y$ be smooth proper varieties of dimension $m$ over $F$ and let $X'$, $Y'$ be smooth proper varieties of dimension $n$ over $F$. If $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow X' \times Y'$, are irreducible correspondences, one checks immediately (using [**GDb**, Proposition 4.2.4]) that $\alpha \times \beta$ can be viewed as a (not necessarily irreducible) correspondence from $X \times X'$ to $Y \times Y'$. This construction generalizes in the obvious way to general correspondences $\alpha$ and $\beta$.

Suppose also that all of these varieties are cohomologically torsion-free at $l$. In this situation we have natural Künneth projections fitting into a commutative diagram

$$\begin{array}{ccc} H^{i+j}\big(X_{\bar F} \times X'_{\bar F}, \mathbf{Z}_l(a+b)\big) & \longrightarrow & H^i\big(X_{\bar F}, \mathbf{Z}_l(a)\big) \otimes_{\mathbf{Z}_l} H^j\big(X'_{\bar F}, \mathbf{Z}_l(b)\big) \\ \big\downarrow {\scriptstyle (\alpha\times\beta)_*} & & \big\downarrow {\scriptstyle \alpha_*\times\beta_*} \\ H^{i+J}\big(Y_{\bar F} \times Y'_{\bar F}, \mathbf{Z}_l(a+b)\big) & \longrightarrow & H^i\big(Y_{\bar F}, \mathbf{Z}_l(a)\big) \otimes_{\mathbf{Z}_l} H^j\big(Y'_{\bar F}, \mathbf{Z}_l(b)\big) \end{array} \tag{2.1}$$

for any $i, j \geq 0$ and any integers $a, b$. (See [**FK88**, Chapter 1, Corollay 8.17] for information on the Künneth theorem.) The commutativity of (2.1) follows immediately from the compatibility of Künneth projections with maps coming from finite, flat morphisms; indeed, its inverse comes from cup product and maps on cohomology induced by various projections, and these are all appropriately functorial. The corresponding diagram for $(\alpha \times \beta)^*$ commutes for the same reason. Of course, this commutativity is true for far more general étale sheaves than twists of $\mathbf{Z}_l$; we state it this way purely for notational reasons.

If the varieties are not all cohomologically torsion-free, we can still define Künneth projections and a diagram analogous to (2.1) provided that we work with $\mathbf{Q}_l$-coefficients rather than $\mathbf{Z}_l$-coefficients.

**2.2. Poincaré duality.** Let $X$ and $Y$ be smooth proper varieties of dimension $n$ over $F$ and let $\alpha$ be an irreducible correspondence from $X$ to $Y$. Let

$$\varphi_X : H^i(X, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^{2n-i}(X, \mathbf{Z}_l) \to \mathbf{Z}_l(-n)$$
$$\varphi_Y : H^i(Y, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^{2n-i}(Y, \mathbf{Z}_l) \to \mathbf{Z}_l(-n)$$

be the Poincaré pairings for some $i \geq 0$; see [**FK88**, Chapter 2, Section 1]. These pairings are compatible in the sense that

$$\varphi_X(h, \alpha^* h') = \varphi_Y(\alpha_* h, h') \tag{2.2}$$

for $h \in H^i(X, \mathbf{Z}_l)$ and $h' \in H^{2n-i}(Y, \mathbf{Z}_l)$; and

$$\varphi_X(\alpha^* h, h') = \varphi_Y(h, \alpha_* h') \tag{2.3}$$

for $h \in H^i(Y, \mathbf{Z}_l)$ and $h' \in H^{2n-i}(X, \mathbf{Z}_l)$.

(2.2) follows immediately from the commutative diagram

$$\begin{array}{ccccc} H^i\big(X, \mathbf{Z}_l(a)\big) & \otimes_{\mathbf{Z}_l} & H^{2n-i}\big(X, \mathbf{Z}_l(b)\big) & \longrightarrow & H^{2n}\big(X, \mathbf{Z}_l(a+b)\big) \\ {\scriptstyle \pi_X^*}\big\downarrow & & \big\uparrow {\scriptstyle \pi_{X*}} & & \big\uparrow {\scriptstyle \pi_{X*}} \\ H^i\big(\alpha, \mathbf{Z}_l(a)\big) & \otimes_{\mathbf{Z}_l} & H^{2n-i}\big(\alpha, \mathbf{Z}_l(b)\big) & \longrightarrow & H^{2n}\big(\alpha, \mathbf{Z}_l(a+b)\big) \\ {\scriptstyle \pi_{Y*}}\big\downarrow & & \big\uparrow {\scriptstyle \pi_Y^*} & & \big\downarrow {\scriptstyle \pi_{Y*}} \\ H^i\big(Y, \mathbf{Z}_l(a)\big) & \otimes_{\mathbf{Z}_l} & H^{2n-i}\big(Y, \mathbf{Z}_l(b)\big) & \longrightarrow & H^{2n}\big(Y, \mathbf{Z}_l(a+b)\big) \end{array}$$

(where the horizontal maps are cup product) and the fact that $\pi_{X*}$ and $\pi_{Y*}$ induce the canonical isomorphisms on the top cohomology groups. The proof of (2.3) is similar.

**2.3. The Flach map.** Let $X$ be as above. Note that the purity hypothesis required to define the Flach map on $X$are automatic since $F$ is perfect. For any irreducible correspondence $\alpha \hookrightarrow X \times Y$ there are commutative diagrams

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} & \xrightarrow{\alpha_*} & E_2^{m,-m-1}(Y)_{0,\mathbf{Z}_l} \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle \sigma_m} \\ H^1\big(F, H^{2m}(X_{\bar{F}}, \mathbf{Z}_l(m+1))\big) & \xrightarrow{\alpha_*} & H^1\big(F, H^{2m}(Y_{\bar{F}}, \mathbf{Z}_l(m+1))\big) \end{array}$$

$$\begin{array}{ccc} E_2^{m,-m-1}(Y)_{0,\mathbf{Z}_l} & \xrightarrow{\alpha^*} & E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \\ \downarrow{\scriptstyle \sigma_m} & & \downarrow{\scriptstyle \sigma_m} \\ H^1\big(F, H^{2m}(Y_{\bar{F}}, \mathbf{Z}_l(m+1))\big) & \xrightarrow{\alpha^*} & H^1\big(F, H^{2m}(X_{\bar{F}}, \mathbf{Z}_l(m+1))\big) \end{array}$$

The commutativity of these diagrams follows immediately from the compatibility of the Flach map with pullback under flat morphisms and trace maps under finite, flat morphisms; see Section VI.7.

## 3. Composition of correspondences

Let $X, Y, Z$ be smooth proper varieties of dimension $n$ over $F$. Let $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow Y \times Z$ be irreducible correspondences. Under certain circumstances we will define the composition $\beta \circ \alpha$ as a correspondence from $X$ to $Z$.

Begin by considering the scheme-theoretic intersection

$$\Gamma = (\alpha \times Z) \cap (X \times \beta) \hookrightarrow X \times Y \times Z.$$

Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of $\Gamma$. Each has dimension at least $n$; we will see in a moment that each in fact has dimension exactly $n$.

Lemma 3.1. *Each irreducible component $\Gamma_i$ is generically reduced.*

Proof. Let $A$ and $B$ be smooth open subsets of irreducible components of $\alpha \times Z$ and $X \times \beta$ respectively. Further shrink $A$ so that the projection $A \to Y$ is smooth; we can do this by [**GDb**, Corollaries 6.12.5 and 17.15.2], using the fact that $F$ is perfect. To prove the lemma it will suffice to show that $A$ and $B$ intersect transversally at all geometric points; see for example [**Ful98**, Section 8.2, esp. Remark 8.2] and [**GDb**, Definition 17.13.3 and Proposition 17.13.8].

Let $c$ be a geometric point of $A \cap B$. Since $X \times Y \times Z$ is smooth of dimension $3n$, the tangent space $T_c(X \times Y \times Z)$ has dimension $3n$ over $\bar{F}$ and has a canonical direct sum decomposition as $T_cX \oplus T_cY \oplus T_cZ$. The tangent spaces $T_cA$ and $T_cB$ are both $2n$-dimensional, since $A$ and $B$ are smooth of dimension $2n$ at $c$. Clearly by our construction we have canonical injections $T_cX \hookrightarrow T_cB$ and $T_cZ \hookrightarrow T_cA$.

For $A$ and $B$ to intersect non-transversally at $c$ means precisely that $T_cA \cap T_cB$ has dimension greater than $n$. In particular, if this is the case then $T_cA$ must have non-trivial intersection with $T_cX$. Since already $T_cZ$ injects into $T_cA$, it follows that the projection $T_cA \to T_cY$ is not surjective. Since the map $A \to Y$ was assumed to be smooth at $c$ this contradicts [**GDb**, Theorem 17.11.1] and completes the proof. ☐

Let $\gamma \hookrightarrow X \times Z$ be the scheme-theoretic image of $\Gamma$ under the projection $\pi_{XZ} : X \times Y \times Z \to X \times Z$ and let $\gamma_i$ be the scheme-theoretic image of $\Gamma_i$. Each $\gamma_i$ is irreducible and generically reduced by Lemma 3.1 and [**GDc**, Proposition 9.5.9].

LEMMA 3.2. *For each $i$, the projections $\gamma_i \to X$ and $\gamma_i \to Z$ are finite and surjective.*

PROOF. We first show that $\gamma \to X$ is quasi-finite and surjective. Since everything in sight is finite type over a perfect field, in both cases it is enough to work on the level of geometric points; see [**GDb**, Proposition 9.3.2, Corollaries 10.4.8 and 13.1.4], although what we are using is really much easier. Let $x \in X(\bar{F})$ be an arbitrary geometric point. Since the map $\alpha \to X$ is finite and surjective, there is a finite non-empty set of points $(x, y_1), \dots, (x, y_d)$ in the fiber over $x$. Since $\beta \to Y$ is also finite and surjective, for each $y_i$ there is a finite non-empty set of points $(y_i, z_{i1}), \dots, (y_i, z_{ie_i})$ in the fiber over $y_i$. Thus the fiber over $x$ in $\Gamma$ is precisely the finite non-empty set of points $(x, y_i, z_{ij})$ and the fiber over $x$ in $\gamma$ consists of the points $(x, z_{ij})$. Thus $\gamma \to X$ is quasi-finite and surjective. This also shows that $\Gamma \to X$ and $\Gamma \to \gamma$ are quasi-finite and surjective; in particular $\Gamma$ has dimension $n$ since $X$ does. Since each $\Gamma_i$ has dimension at least $n$, it follows that they all have dimension exactly $n$.

By base change we also see that each $\Gamma_i \to \gamma_i$ is quasi-finite and surjective. In particular, each $\gamma_i$ has dimension exactly $n$. $\gamma$ is a closed subscheme of $X \times Z$ and thus is proper over $X$. Since quasi-finite and proper imply finite [**GDa**, Proposition 4.4.2], we conclude that the projection $\gamma \to X$ is finite and surjective.

Now consider the projection $\gamma_i \to X$ of an irreducible component of $\gamma$. This is the composition of the closed immersion $\gamma_i \to \gamma$ with the finite map $\gamma \to X$, and thus is finite. In particular, it is also proper, so the image is a closed subset of $X$. Since $X$ is irreducible, if this image were not all of $X$, then it would have smaller dimension; since $\gamma_i \to X$ is a finite map of schemes of the same dimension, this is impossible. Thus $\gamma_i \to X$ is surjective. (See also [**GDb**, Proposition 5.4.1(ii)].) The proof for $\gamma_i \to Z$ is identical. ☐

Given all of this, we define the composition $\beta \circ \alpha$ only under the additional assumption:

- The projections $\gamma_i \to X$ and $\gamma_i \to Z$ are flat for all $i$;

With these hypotheses, we define $\gamma = \beta \circ \alpha$ as

$$\sum m_i \gamma_i$$

where $m_i = [k(\Gamma_i) : k(\gamma_i)]$. This makes sense since by Lemma 3.2 and the assumption above the maps $\gamma_i \to X, Z$ are finite and faithfully flat. (The fact that the $\gamma_i$ are generically reduced means that we need not introduce any multiplicities back on $\Gamma$.)

If $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ are correspondences such that each composition $\beta_j \circ \alpha_i$ is defined, we define the composition of $\sum \alpha_i$ and $\sum \beta_j$ in the obvious way.

## 4. Marked varieties

Fix integers $n$, $k$, $w$ such that $k = n^w$. For any $F$-scheme $X$, define

$$\mathcal{L}_X = \wedge^k \left(\Omega_{X/F}^{\otimes w}\right);$$

$\mathcal{L}_X$ is always to be considered as a Zariski sheaf, not an étale sheaf. The construction of $\mathcal{L}_X$ is functorial, in the sense that if there is a map $f : X \to Y$ over $\operatorname{Spec} F$, then there is an induced map $f^*\mathcal{L}_Y \to \mathcal{L}_X$ of Zariski sheaves on $X$; this is immediate from the functoriality of sheaves of differentials. If $X$ is a smooth variety of dimension $n$ over $F$, then $\mathcal{L}_X$ is an invertible sheaf by our choice of $k$ and $w$.

Definition 4.1. A *marking* $\omega_X$ on a smooth $F$-scheme $X$ of dimension $n$ is a non-zero rational section of $\mathcal{L}_X$. That is, a marking is an equivalence class of pairs of a dense open set $U \subseteq X$ and a non-zero section $\omega \in \mathcal{L}_X(U)$.

We should note that which sheaf we use here is not particularly important; one could replace $\mathcal{L}_X$ by any other functorial Zariski sheaf which is invertible on the smooth locus of $F$-schemes of dimension $n$.

Now let $X$ and $Y$ be smooth proper varieties of dimension $n$ over $\operatorname{Spec} F$. Let $\omega_X$ and $\omega_Y$ be markings on $X$ and $Y$ and let $\alpha \hookrightarrow X \times Y$ be an irreducible correspondence from $X$ to $Y$. We will use the markings on $X$ and $Y$ to define a rational function $f_\alpha = f_\alpha(\omega_X, \omega_Y)$ on $\alpha$; we will always assume that $\omega_X$ and $\omega_Y$ are fixed for the discussion and suppress them from the notation.

The definition of $f_\alpha$ is as follows: let $U_X$ and $U_Y$ be open sets on which $\omega_X$ and $\omega_Y$ are defined, respectively. Let $V$ be an open subset of $\alpha$ contained in the intersection of $\pi_X^{-1}(U_X)$, $\pi_Y^{-1}(U_Y)$ and the smooth locus of $\alpha$; further shrink $V$ so that $\mathcal{L}_\alpha$ is free (necessarily of rank 1) over $V$. Since $F$ is perfect, such $V$ exist by [**GDb**, Corollaries 6.12.5 and 17.15.2]. $\pi_X$ is flat and thus open, so $\pi_X(V)$ and $\pi_Y(V)$ are open subsets of $U_X$ and $U_Y$ respectively. Evaluating the map of sheaves $\pi_X^*\mathcal{L}_X \to \mathcal{L}_\alpha$ at $V$ and composing with appropriate restriction maps we obtain a map

$$\mathcal{L}_X(U_X) \to \mathcal{L}_\alpha(V). \tag{4.1}$$

We denote by $\pi_X^*\omega_X$ the image of $\omega_X$ under (4.1), viewed as a rational section of $\mathcal{L}_\alpha$. We define $\pi_Y^*\omega_Y$ similarly. The rational function $f_\alpha \in k(\alpha)^\times$ is now simply the ratio

$$\frac{\pi_X^*\omega_X}{\pi_Y^*\omega_Y} \in k(\alpha)^\times; \tag{4.2}$$

this makes sense as $\mathcal{L}_\alpha|_V$ is free of rank 1 over $\mathcal{O}_V$, and it is clear that (4.2) is independent of the choices of $U_X$, $U_Y$ and $V$. $f_\alpha$ is non-zero and "not infinite" since $\omega_X, \omega_Y$ are non-zero and $\pi_X, \pi_Y$ are finite and surjective.

If $\alpha = \sum m_i\alpha_i$ is a general correspondence, we use the markings on $X$ and $Y$ to associate to $\alpha$ the rational function $f_\alpha$ on $\alpha$ given by $f_{\alpha_i}^{m_i}$ on $\alpha_i$.

We view the pair $(\alpha, f_\alpha)$ as an element of the spectral sequence $E_1^{n,-n-1}(X\times Y)$; if $\alpha = \sum m_i\alpha_i$ is not irreducible then we view it as the element $\sum(\alpha_i, f_{\alpha_i}^{m_i})$ of $E_1^{n,-n-1}(X \times Y)$ in the usual way.

Definition 4.2. Let $\alpha$ be an algebraic correspondence from $X$ to $Y$. We will say that $\alpha$ is *admissible* for the given markings $\omega_X, \omega_Y$ on $X$ and $Y$ if the Weil divisor of $f_\alpha$ is trivial on $\alpha$.

Here by the Weil divisor of $f_\alpha$ we mean the sum of the Weil divisors on the irreducible component; in particular, we allow these to be non-trivial so long as they cancel each other out. A similar effect can occur if $\alpha$ is irreducible but singular.

If $\alpha$ is an admissible correspondence, then by (VI.5.3) $(\alpha, f_\alpha)$ defines an element of $E_2^{n,-n-1}(X \times Y)$. By Lemma VI.6.2 we can define a Flach class (depending also on $\omega_X$ and $\omega_Y$)

$$\sigma_{X,Y}(\alpha) = \sigma_n(\alpha, f_\alpha) \in H^1\big(F, H^{2n}(X_{\bar F} \times Y_{\bar F}, \mathbf{Q}_l(n+1))\big). \tag{4.3}$$

If $H^{2n+1}(X_{\bar F} \times Y_{\bar F}, \mathbf{Z}_l)$ is torsion-free, then we can even realize this class as

$$\sigma_{X,Y}(\alpha) = \sigma_n(\alpha, f_\alpha) \in H^1\big(F, H^{2n}(X_{\bar F} \times Y_{\bar F}, \mathbf{Z}_l(n+1))\big). \tag{4.4}$$

Note that in order to make this construction it does not seem to be enough to know $\alpha$ up to rational equivalence; this is why we are forced to use the somewhat restricted definition of correspondence which we are using.

## 5. Divisors and compositions

Let $X, Y, Z$ be smooth proper varieties of dimension $n$ over $F$. Let $\omega_X, \omega_Y, \omega_Z$ be markings on $X, Y, Z$ (for some fixed $k, w$ as in Section 4) and let $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow Y \times Z$ be irreducible correspondences. Suppose also that the composition $\gamma = \beta \circ \alpha$ is defined as a correspondence from $X$ to $Z$; let $\gamma_0$ be an irreducible component. The markings determine rational functions $f_\alpha$, $f_\beta$, $f_{\gamma_0}$ on $\alpha$, $\beta$, $\gamma_0$ respectively. We want to relate the admissibility condition on $\alpha$ and $\beta$ to that on $\gamma_0$.

For this result we will need to use pullbacks of divisors by finite, surjective maps. That is, given a finite, surjective map $\pi : X \to Y$ and a codimension 1 cycle $Z$ on $Y$, we define $\pi^* Z$ to be the cycle class (in the sense of [**Ful98**, Section 1.3]) of $\pi^{-1} Z = X \times_Y Z$. One checks easily from the fact that $\pi$ is integral that every irreducible component of $\pi^{-1} Z$ has codimension 1 in $X$.

There is not a particularly good theory of such pullbacks (for example, they may not respect rational equivalence), but they do satisfy the following two properties which will be sufficient for our purposes: First, the composition $\pi_* \pi^*$ of the pullback with the proper pushforward is injective on the free abelian group of codimension 1 cycles; indeed, it sends any cycle $Z$ to a non-zero multiple of itself, from which this injectivity follows immediately. Second, if $\pi' : Y \to Y'$ is a finite, flat morphism, then the finite surjective pullback $(\pi' \pi)^*$ is the same as the composition $\pi^* \pi'^*$; here $\pi'^*$ is the usual intersection theoretic pullback.

LEMMA 5.1. *With the above notation, suppose also that* $\operatorname{div}_\alpha f_\alpha = \operatorname{div}_\beta f_\beta = 0$. *Then* $\operatorname{div}_{\gamma_0} f_{\gamma_0} = 0$.

PROOF. Let $\Gamma_0$ be the irreducible component of $\Gamma = (\alpha \times Z) \cap (X \times \beta)$ mapping to $\gamma_0$. Note that the projection $\pi_{\Gamma_0 X} : \Gamma_0 \to X$ factors through the map $\pi_{\alpha X} : \alpha \to X$. Similarly, the projection $\pi_{\Gamma_0 Y}$ factors through both $\pi_{\alpha Y}$ and $\pi_{\beta Y}$, and the projection $\pi_{\Gamma_0 Z}$ factors through $\pi_{\beta Z}$.

The statement that $\operatorname{div}_\alpha f_\alpha = 0$ is precisely the statement that $\operatorname{div}_\alpha \pi_{\alpha X}^* \omega_X = \operatorname{div}_\alpha \pi_{\alpha Y}^* \omega_Y$. By compatibility of finite, flat pullback with divisors, this is the same as the equality

$$\pi_{\alpha X}^* \operatorname{div}_X \omega_X = \pi_{\alpha Y}^* \operatorname{div}_Y \omega_Y. \tag{5.1}$$

Pulling back (5.1) by the finite, surjective morphism $\pi_{\Gamma_0\alpha}$ yields

$$\pi^*_{\Gamma_0 X} \operatorname{div}_X \omega_X = \pi^*_{\Gamma_0 Y} \operatorname{div}_Y \omega_Y.$$

Using the same sort of argument for $\beta$, we conclude that

$$\pi^*_{\Gamma_0 X} \operatorname{div}_X \omega_X = \pi^*_{\Gamma_0 Z} \operatorname{div}_Z \omega_Z.$$

Applying the proper pushforward $\pi_{\Gamma_0\gamma_0 *}$ to this and using the functoriality of finite, surjective pullbacks with flat pullbacks, we find that

$$\pi_{\Gamma_0\gamma_0 *}\pi^*_{\Gamma_0\gamma_0}\pi^*_{\gamma_0 X} \operatorname{div}_X \omega_X = \pi_{\Gamma_0\gamma_0 *}\pi^*_{\Gamma_0\gamma_0}\pi^*_{\gamma_0 Z} \operatorname{div}_Z \omega_Z. \tag{5.2}$$

Since $\pi_{\Gamma_0\gamma_0 *}\pi^*_{\Gamma_0\gamma_0}$ is injective, we conclude from (5.2) that

$$\pi^*_{\gamma_0 X} \operatorname{div}_X \omega_X = \pi^*_{\gamma_0 Z} \operatorname{div}_Z \omega_Z.$$

Compatibility of flat pullbacks with divisors now yields the desired equality. □

## 6. The Leibniz relation

We keep the notation of the previous section. Further assume that $\alpha$ and $\beta$ are admissible; Lemma 5.1 insures that $\gamma = \beta \circ \alpha$ is as well. Assuming that $X$, $Y$ and $Z$ are cohomologically torsion-free at $l$, we can define Flach classes

$$\begin{aligned} \sigma_{X,Y}(\alpha) &\in H^1\big(F, H^{2n}(X_{\bar F} \times Y_{\bar F}, \mathbf{Z}_l(n+1))\big); \\ \sigma_{Y,Z}(\beta) &\in H^1\big(F, H^{2n}(Y_{\bar F} \times Z_{\bar F}, \mathbf{Z}_l(n+1))\big); \\ \sigma_{X,Z}(\gamma) &\in H^1\big(F, H^{2n}(X_{\bar F} \times Z_{\bar F}, \mathbf{Z}_l(n+1))\big) \end{aligned} \tag{6.1}$$

as in (4.4); even if the groups are not torsion-free, we can still define these classes after tensoring with $\mathbf{Q}_l$ as in (4.3). These classes are related by the following beautiful formula of Mazur and Beilinson.

We will first need some notation. Let $\Delta_Z \hookrightarrow Z \times Z$ be the diagonal viewed as an algebraic correspondence from $Z$ to $Z$; both $\Delta_*$ and $\Delta^*$ are the identity map on $K$-theory and étale cohomology. View $\alpha \times \Delta_Z \hookrightarrow X \times Y \times Z \times Z$ as a correspondence from $X \times Z$ to $Y \times Z$; one checks immediately that it satisfies the required hypotheses. Similarly, view $\Delta_X \times \beta \hookrightarrow X \times X \times Y \times Z$ as a correspondence from $X \times Y$ to $X \times Z$. Recall that we can also use maps coming from correspondences to yield maps on Galois cohomology. We will consider the induced maps

$$(\alpha \times \Delta_Z)^* : H^1\big(F, H^{2n}(Y_{\bar F} \times Z_{\bar F}, \mathbf{Z}_l(n+1))\big) \to H^1\big(F, H^{2n}(X_{\bar F} \times Z_{\bar F}, \mathbf{Z}_l(n+1))\big)$$

$$(\Delta_X \times \beta)^* : H^1\big(F, H^{2n}(X_{\bar F} \times Y_{\bar F}, \mathbf{Z}_l(n+1))\big) \to H^1\big(F, H^{2n}(X_{\bar F} \times Z_{\bar F}, \mathbf{Z}_l(n+1))\big).$$

THEOREM 6.1. *Let $\alpha$ be a correspondence from $X$ to $Y$ and let $\beta$ be a correspondence from $Y$ to $Z$. Assume that $\gamma = \beta \circ \alpha$ is defined as a correspondence from $X$ to $Z$ and that $\alpha$ and $\beta$ are admissible for our fixed choice of markings. If all of the integral Flach classes (6.1) are defined then*

$$\sigma_{X,Z}(\gamma) = (\alpha \times \Delta_Z)^* \sigma_{Y,Z}(\beta) + (\Delta_X \times \beta)_* \sigma_{X,Y}(\alpha). \tag{6.2}$$

*If the integral Flach classes are not defined, then this formula still holds after tensoring with $\mathbf{Q}_l$ as in (4.3).*

PROOF. By linearity we can assume that $\alpha$ and $\beta$ are irreducible. We first prove the formula on the level of algebraic cycles and $K$-theory. That is, we wish to show that in $E_1^{n,-n-1}(X \times Z)$ we have the equality

$$(\gamma, f_\gamma) = (\alpha \times \Delta_Z)^*(\beta, f_\beta) + (\Delta_X \times \beta)_*(\alpha, f_\alpha). \tag{6.3}$$

Consider first $(\alpha \times \Delta_Z)^*(\beta, f_\beta)$. The "cycle" part of this is obtained as follows: one pulls back and pushes forward $\beta \hookrightarrow Y \times Z$ in the diagram

$$\begin{array}{ccc} & \alpha \times \Delta_Z & \\ \swarrow & & \searrow \\ Y \times Z & & X \times Z \end{array}$$

Let $\beta'$ be the image of $\beta$ under the map $\mathrm{id} \times \Delta : Y \times Z \to Y \times Z \times Z$. Pulling back $\beta$ to $\alpha \times \Delta_Z$ is the same as forming the scheme-theoretic intersection

$$(X \times \beta') \cap (\alpha \times \Delta_Z) \hookrightarrow X \times Y \times Z \times Z. \tag{6.4}$$

The projection from here to $X \times Z$ factors through $X \times Y \times Z$; here the image of (6.4) is just the intersection of $X \times \beta$ and $\alpha \times Z$. In particular, by our definition of composition of correspondences the final image of $\beta$ in $X \times Z$ is nothing other than $\beta \circ \alpha = \gamma$.

Since $f_\beta$ is $\pi^*_{\beta Y}\omega_Y / \pi^*_{\beta Z}\omega_Z$, tracing through the maps we see that the corresponding rational function on an irreducible component $\gamma_i$ of $\gamma$ is

$$\frac{N_{k(\Gamma_i)/k(\gamma_i)}\pi^*_{\Gamma_i Y}\omega_Y}{\pi^*_{\gamma_i Z}\omega_Z^{m_i}}$$

where $\Gamma_i$ is the irreducible component of $\Gamma$ surjecting onto $\gamma_i$ and $m_i = [k(\Gamma_i) : k(\gamma_i)]$. That is, writing $\gamma = \sum m_i \gamma_i$ as a sum of irreducible correspondences, we have

$$(\alpha \times \Delta_Z)^*(\beta, f_\beta) = \sum \left( \gamma_i, \frac{N_{k(\Gamma_i)/k(\gamma_i)}\pi^*_{\Gamma_i Y}\omega_Y}{\pi^*_{\gamma_i Z}\omega_Z^{m_i}} \right). \tag{6.5}$$

Similarly, we have

$$(\Delta_X \times \beta)_*(\alpha, f_\alpha) = \sum \left( \gamma_i, \frac{\pi^*_{\gamma_i X}\omega_X^{m_i}}{N_{k(\Gamma_i)/k(\gamma_i)}\pi^*_{\gamma_i Y}\omega_Y} \right). \tag{6.6}$$

Adding (6.5) and (6.6) in $E_1^{n,-n-1}(X \times Z)$ yields

$$(\alpha \times \Delta_Z)^*(\beta, f_\beta) + (\Delta_X \times \beta)_*(\alpha, f_\alpha) = \sum \left( \gamma_i, \left( \frac{\pi^*_{\gamma_i X}\omega_X}{\pi^*_{\gamma_i Z}\omega_Z} \right)^{m_i} \right)$$

which is precisely the element $(\gamma, f_\gamma)$.

Since $\alpha$, $\beta$ and $\gamma$ are all admissible, the equality (6.3) in $E_1^{n,-n-1}(X \times Z)$ yields the same equality in $E_2^{n,-n-1}(X \times Z)$. The fact that (6.2) holds in étale cohomology now follows immediately from the compatibility of the Flach map with maps coming from correspondences as in Section 2. $\square$

Assume now that $X$, $Y$ and $Z$ are all cohomologically torsion-free at $l$. In this situation we have Künneth projections on étale cohomology as in Section 2. Let $\sigma'_{X,Y}(\alpha)$ denote the image of $\sigma_{X,Y}(\alpha)$ under the map

$$H^1\big(F, H^{2n}(X_{\bar{F}} \times Y_{\bar{F}}, \mathbf{Z}_l(n+1))\big) \to H^1\big(F, H^n(X_{\bar{F}}, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^n(Y_{\bar{F}}, \mathbf{Z}_l)(n+1)\big)$$

induced by the Künneth projection; we define $\sigma'_{Y,Z}(\beta)$ and $\sigma'_{X,Z}(\gamma)$ similarly. By the compatibility of correspondences with Künneth projections, we see that (6.2) now takes the form

$$\sigma'_{X,Z}(\gamma) = (\alpha^* \otimes 1)\sigma'_{Y,Z}(\beta) + (1 \otimes \beta_*)\sigma'_{X,Y}(\alpha). \tag{6.7}$$

We are again using $\alpha^*$ and $\beta_*$ to induce maps on Galois cohomology:

$$\alpha^* : H^n(Y_{\bar{F}}, \mathbf{Z}_l) \to H^n(X_{\bar{F}}, \mathbf{Z}_l)$$

$$\beta_* : H^n(Y_{\bar{F}}, \mathbf{Z}_l) \to H^n(Z_{\bar{F}}, \mathbf{Z}_l).$$

As usual, we can obtain analogous results after tensoring by $\mathbf{Q}_l$ even if the varieties are not all cohomologically torsion-free.

## 7. Algebras of correspondences

Let $X$ be a smooth proper variety of dimension $n$ over $F$. By an *algebra of correspondences* on $X$ we will mean a set $\mathcal{A}$ of correspondences from $X$ to $X$ which forms a (possibly infinitely generated and non-commutative) $\mathbf{Z}$-algebra with composition of correspondences as multiplication. In particular, it is assumed that every composition of elements of $\mathcal{A}$ is defined. We assume that $\Delta_X$ lies in $\mathcal{A}$; it serves as a multiplicative identity element.

We say that a marking $\omega_X$ is *admissible* for $\mathcal{A}_0$ if every $\alpha \in \mathcal{A}_0$ is admissible for $\omega_X$. Note that to check that an algebra $\mathcal{A}$ is admissible for a given marking, by Lemma 5.1 it suffices to check on a set of algebra generators of $\mathcal{A}$.

Now let $\mathcal{A}_0$ be an algebra of correspondences on $X$ and let $\mathcal{A} = \mathcal{A}_0 \otimes_{\mathbf{Z}} \mathbf{Z}_l$ for some fixed prime $l$; $\mathcal{A}$ is a (possibly infinitely generated and non-commutative) $\mathbf{Z}_l$-algebra. For any fixed $m$, $\mathcal{A}$ admits two maps to $\operatorname{End}_{\mathbf{Z}_l} H^m(X_{\bar{F}}, \mathbf{Z}_l)$, one given by $\alpha \mapsto \alpha_*$ and one given by $\alpha \mapsto \alpha^*$.

Let $\omega_X$ be an admissible marking for $\mathcal{A}_0$. Assume also that $X$ is cohomologically torsion-free at $l$. We write the map $\sigma_{X,X}$ of (4.3) as $\sigma$; we consider it as a map

$$\sigma : \mathcal{A} \to E_2^{n,-n-1}(X \times X) \otimes_{\mathbf{Z}} \mathbf{Z}_l \to H^1\big(F, H^{2n}(X_{\bar{F}} \times X_{\bar{F}}, \mathbf{Z}_l(n+1))\big)$$

sending $\alpha$ to $\sigma_{X,X}(\alpha, f_\alpha)$. We now apply the Künneth analysis at the end of the previous section. Specifically, let $V = H^n(X_{\bar{F}}, \mathbf{Z}_l)$ and let

$$\tau : \mathcal{A} \to H^1\big(F, V \otimes_{\mathbf{Z}_l} V(n+1)\big)$$

denote the composition of $\sigma$ with the map

$$H^1\big(F, H^{2n}(X_{\bar{F}} \times X_{\bar{F}}, \mathbf{Z}_l(n+1))\big) \to H^1\big(F, H^n(X_{\bar{F}}, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^n(X_{\bar{F}}, \mathbf{Z}_l)(n+1)\big)$$

coming from the Künneth projection. The Leibniz relation (6.7) takes the form

$$\tau(\beta\alpha) = (\alpha^* \otimes 1)\tau(\beta) + (1 \otimes \beta_*)\tau(\alpha). \tag{7.1}$$

As always we can obtain the same formula over $\mathbf{Q}_l$ without the cohomologically torsion-free hypothesis. For the remainder of the chapter we will assume that $X$ is cohomologically torsion-free; however all results remain true over $\mathbf{Q}_l$ even without this hypothesis. We will not comment on this further.

## 8. Derivations in the self-adjoint case

We keep the hypotheses of the previous section: $X$ is a smooth proper variety of dimension $n$ over $F$ and $\mathcal{A}$ is a $\mathbf{Z}_l$-algebra of self-correspondences on $X$ with an admissible marking $\omega_X$. We assume that $X$ is cohomologically torsion-free at $l$. Set $V = H^n(X_{\bar{F}}, \mathbf{Z}_l)$; we have a map

$$\tau : \mathcal{A} \to H^1\big(F, V \otimes_{\mathbf{Z}_l} V(n+1)\big).$$

The maps $\alpha \to \alpha_*$ and $\alpha \to \alpha^*$ yield two maps $\mathcal{A} \to \operatorname{End}_{\mathbf{Z}_l} V$. Let $B_*$ and $B^*$ denote their images; they are finite, flat $\mathbf{Z}_l$-algebras since $\operatorname{End}_{\mathbf{Z}_l} V$ is. For this section we make the following assumptions:

- $\mathcal{A}$ is commutative;
- $\mathcal{A}$ is *self-adjoint* in the sense that the two maps $\mathcal{A} \to \operatorname{End}_{\mathbf{Z}_l} V$ coincide;

None of these assumptions will actually be used in this section, but if they are not satisfied then the constructions here are not appropriate. We will discuss the elimination of the self-adjoint hypothesis in later sections. For now, we write $B$ for the image of $\mathcal{A}$ in $\operatorname{End}_{\mathbf{Z}_l} V$; $\mathcal{A}$ acts on $V$ in a canonical way via $B$.

In this situation, the functional equation (7.1) for the map $\tau$ can be unambiguously written as

$$\tau(\beta\alpha) = (\alpha \otimes 1)\tau(\beta) + (1 \otimes \beta)\tau(\alpha).$$

That is, $\tau$ is a bilateral derivation in the sense of Section A.6.

We wish to pass from the bilateral derivation $\tau$ to bilateral derivations and derivations to the Galois cohomology of certain quotients of $V \otimes_{\mathbf{Z}_l} V(n+1)$. In the self-adjoint case, this is straightforward. Let $\mathfrak{m}$ be a maximal ideal of $B$ and let $A$ denote the completion of $B$ at $\mathfrak{m}$. $A$ is a finite, flat, local $\mathbf{Z}_l$-algebra and is canonically a direct summand of $B$. $H = V \otimes_B A$ is therefore canonically a direct summand of $V$; let $i : H \hookrightarrow V$ and $j : V \twoheadrightarrow H$ denote the corresponding maps.

We define a bilateral derivation

$$\mathcal{D} : \mathcal{A} \to H^1\big(F, H \otimes_{\mathbf{Z}_l} H(n+1)\big)$$

as the composition of $\tau$ with the map on cohomology induced by $j \otimes j$. We define a map

$$\partial : \mathcal{A} \to H^1\big(F, H \otimes_A H(n+1)\big)$$

as the composition of $\mathcal{D}$ with the map on cohomology induced by the natural surjection

$$H \otimes_{\mathbf{Z}_l} H(n+1) \twoheadrightarrow H \otimes_A H(n+1).$$

Since $\mathcal{D}$ satisfies

$$\mathcal{D}(\beta\alpha) = (\alpha \otimes 1)\mathcal{D}(\beta) + (1 \otimes \beta)\mathcal{D}(\alpha),$$

and the $A \otimes_{\mathbf{Z}_l} A$ action on $H^1(F, H \otimes_A H(n+1))$ factors through the diagonal map $A \otimes_{\mathbf{Z}_l} A \to A$, we see that $\partial$ satisfies

$$\partial(\beta\alpha) = \alpha\partial(\beta) + \beta\partial(\alpha);$$

that is, $\partial$ is a derivation.

## 9. Local diagrams in the self-adjoint case

In the applications of our constructions it is often more convenient the cohomology of $\operatorname{End}_A H(1)$ than $H \otimes_A H(n+1)$. In this section we explain how to make the transition; it is also useful for computational purposes.

For the remainder of this chapter, for any $\mathbf{Z}_l$-module $M$ we denote by $M^\dagger$ its integral Pontrjagin dual $\operatorname{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l)$. If $\varphi : M \otimes_{\mathbf{Z}_l} N \to \mathbf{Z}_l$ is any pairing, we write $\varphi_r : N \to M^\dagger$ for the induced map.

Central to the transition are various pairings induced by Poincaré duality. The basic Poincaré pairing is a Galois equivariant, perfect pairing

$$\varphi : V \otimes_{\mathbf{Z}_l} V(n) \to \mathbf{Z}_l.$$

Since $\mathcal{A}$ is self-adjoint, $\varphi$ satisfies (see Section 2) $\varphi(bv, v') = \varphi(v, bv')$ for all $b \in B$, $v \in V$ and $v' \in V(n)$; that is, $\varphi$ is $B$-hermitian.

Let $\mathfrak{m}$ be a maximal ideal of $B$ as before, and define a pairing

$$\psi : H \otimes_{\mathbf{Z}_l} H(n) \to \mathbf{Z}_l$$

by $\psi(h, h') = \varphi(ih, ih')$. $\psi$ is an $A$-hermitian, Galois equivariant perfect pairing. (The fact that $\psi$ is perfect is an easy computation using properties of localization.) We have a commutative diagram

(9.1)
$$\begin{array}{ccccc} V \otimes_{\mathbf{Z}_l} V(n) & \xrightarrow{\operatorname{id} \otimes \varphi_r} & V \otimes_{\mathbf{Z}_l} V^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} V \\ \downarrow{\scriptstyle j \otimes j} & & \downarrow{\scriptstyle j \otimes i^\dagger} & & \downarrow{\scriptstyle f \mapsto jfi} \\ H \otimes_{\mathbf{Z}_l} H(n) & \xrightarrow{\operatorname{id} \otimes \psi_r} & H \otimes_{\mathbf{Z}_l} H^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} H \end{array}$$

(To show that (9.1) commutes requires the fact that

$$\varphi(ijh, iv) = \varphi(ijh, v)$$

for all $h \in H$ and $v \in V(n)$; this follows from the fact that both $\varphi$ and $\psi$ are perfect.) All of the maps of (9.1) are Galois equivariant and $B$-linear.

We now introduce the sort of maximal ideals of $B$ which we can use to make the desired translation.

DEFINITION 9.1. A maximal ideal $\mathfrak{m}$ of $B$ is said to be *dualizing* if

- $B_\mathfrak{m}$ is reduced;
- $V_\mathfrak{m}$ is free of rank 2 over $B_\mathfrak{m}$.

By Lemma B.4.1 these conditions imply that $A = B_\mathfrak{m}$ is a Gorenstein $\mathbf{Z}_l$-algebra. Fix a Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$; by Lemma B.3.1 this choice induces an isomorphism $H^\dagger \cong \operatorname{Hom}_A(H, A)$. Furthermore, by Lemma B.4.2 there exists a unique $A$-linear, Galois equivariant perfect pairing $\psi' : H \otimes_A H(n) \to A$ such that $\psi$ factors as

$$H \otimes_{\mathbf{Z}_l} H(n) \longrightarrow H \otimes_A H(n) \xrightarrow{\psi'} A \xrightarrow{\operatorname{tr}} \mathbf{Z}_l.$$

We use these trace identifications to extend (9.1) to

(9.2)
$$\begin{array}{ccccc} H \otimes_{\mathbf{Z}_l} H(n) & \xrightarrow{\operatorname{id} \otimes \psi_r} & H \otimes_{\mathbf{Z}_l} H^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} H \\ \downarrow & & \downarrow & & \downarrow \\ H \otimes_A H(n) & \xrightarrow{\operatorname{id} \otimes \psi'_r} & H \otimes_A \operatorname{Hom}_A(H, A) & \longrightarrow & \operatorname{End}_A H \end{array}$$

Recall that the map $\mathrm{End}_{\mathbf{Z}_l} H \to \mathrm{End}_A H$ has an especially simple description on the submodule $\mathrm{End}_A H$ of $\mathrm{End}_{\mathbf{Z}_l} H$; see Lemma B.3.3.

In any event, we can define the desired derivation

$$\partial' : \mathcal{A} \to H^1\big(F, \mathrm{End}_A H(1)\big)$$

as the composition of $\partial$ with the isomorphism from $H(n+1) \otimes_A H$ to $\mathrm{End}_A H(1)$ coming from the bottom row of (9.2). Note that this isomorphism depends on the choice of Gorenstein trace tr, and thus is canonical only up to an element in $A^\times$.

## 10. Derivations in the general case

In this section we carry out the construction of the previous two sections without the self-adjoint hypothesis. Otherwise we continue with the hypotheses of Section 8. We again define $B_*$ and $B^*$ as the images of $\mathcal{A}$ in $\mathrm{End}_{\mathbf{Z}_l} V$. $V$ has a canonical module structure over $B_*$ and $B^*$, and the Poincaré pairing

$$\varphi : V \otimes_{\mathbf{Z}_l} V(n) \to \mathbf{Z}_l$$

now satisfies

$$\varphi(\alpha_* v, v') = \varphi(v, \alpha^* v');$$
$$\varphi(\alpha^* v, v') = \varphi(v, \alpha_* v').$$

We will need to modify $\varphi$ to obtain a $B^*$-hermitian pairing.

We note in passing that the constructions of these sections can be carried out with $V$ replaced by a direct summand of $H^n(X_{\bar{F}}, \mathbf{Z}_l)$ which is stable under both actions of $\mathcal{A}_0$ and which is self-dual under Poincaré duality. We will not comment further on this.

Definition 10.1. An *untwisting* of $V$ (with respect to $\mathcal{A}$) is a triple $(w, \tilde{B}, \xi)$ of an isomorphism of abelian groups $w : V \to V$ satisfying $w(\alpha_* v) = \alpha^* w(v)$ and $w(\alpha^* v) = \alpha_* w(v)$; a free $B^*$-module of rank 1 $\tilde{B}$ with a $B^*$-linear action of $G_F$; and a chosen generator $\xi$ of $\tilde{B}$ such that the map

$$\xi \otimes w : V \to \tilde{B} \otimes_{B^*} V$$

is Galois equivariant.

Note that the notion of untwisting is actually independent of the choice of generator $\xi$. We include $\xi$ in the notation for simplicity, although our final constructions will not depend on it.

Fix an untwisting $(w, \tilde{B}, \xi)$ and set $\tilde{V} = \tilde{B} \otimes_{B^*} V$. We define a pairing

$$\varphi' : V \otimes_{\mathbf{Z}_l} \tilde{V}(n) \to \mathbf{Z}_l$$

by $\varphi'(v, \xi \otimes v') = \varphi(v, w^{-1} v')$. $\varphi'$ is $B^*$-hermitian and Galois equivariant by the definition of an untwisting.

Define

$$\mathcal{D}_0 : \mathcal{A} \to H^1\big(F, V \otimes_{\mathbf{Z}_l} \tilde{V}(n+1)\big)$$

to be the composition of $\tau$ with the map on cohomology induced by $\mathrm{id} \otimes \xi \otimes w$. We claim that $\mathcal{D}_0$ can be regarded as a bilateral derivation. To check this, let $\alpha, \beta \in \mathcal{A}$ and $\gamma \in G_F$ be any elements, and write

$$\tau(\alpha)(\gamma) = \sum t'_i \otimes t_i$$
$$\tau(\beta)(\gamma) = \sum u'_i \otimes u_i$$

be the evaluation of the cocycles at $\gamma$. We compute

$$\begin{aligned}
\mathcal{D}_0(\beta\alpha)(\gamma) &= (\mathrm{id}\otimes w)\tau(\beta\alpha)(\gamma) \\
&= (\mathrm{id}\otimes w)\big((\alpha^*\otimes 1)\tau(\beta) + (1\otimes\beta_*)\tau(\alpha)\big) \\
&= (\mathrm{id}\otimes w)\left(\sum \alpha^* u_i' \otimes u_i + \sum t_i' \otimes \beta_* t_i\right) \\
&= \sum \alpha^* u_i' \otimes w u_i + \sum t_i' \otimes w\beta_* t_i \\
&= \sum \alpha^* u_i' \otimes w u_i + \sum t_i' \otimes \beta^* w t_i \\
&= (\alpha^*\otimes 1)\mathcal{D}_0(\beta) + (1\otimes\beta^*)\mathcal{D}_0(\alpha).
\end{aligned}$$

Thus $\mathcal{D}_0$ is indeed a bilateral derivation when $V$ and $\tilde{V}$ are given $\mathcal{A}$-module structures via $B^*$.

Now choose a maximal ideal $\mathfrak{m}$ of $B^*$. Let $A = B^*_{\mathfrak{m}}$ and set $\tilde{A} = \tilde{B} \otimes_{B^*} A$; we will also write $\xi$ for the image of $\xi$ in $\tilde{A}$. Set $H = V \otimes_{B^*} A$ and $\tilde{H} = \tilde{A} \otimes_A H$. We have natural maps $i : H \hookrightarrow V$ and $j : V \twoheadrightarrow H$. We define a pairing

$$\psi : H \otimes_{\mathbf{Z}_l} \tilde{H}(n) \to \mathbf{Z}_l$$

by

$$\psi(h, \xi\otimes h') = \varphi'(ih, \xi\otimes ih') = \varphi(ih, w^{-1}ih').$$

$\psi$ is $A$-hermitian and Galois equivariant.

DEFINITION 10.2. A maximal ideal $\mathfrak{m}$ of $B^*$ is said to be *dualizing* if

- $B_{\mathfrak{m}}$ is reduced;
- $V_{\mathfrak{m}}$ is free of rank 2 over $B_{\mathfrak{m}}$.

Fix a dualizing maximal ideal $\mathfrak{m}$. By Lemma B.4.1 $A = B_{\mathfrak{m}}$ is Gorenstein. Let $\mathrm{tr} : A \to \mathbf{Z}_l$ be a choice of Gorenstein trace and let

$$\psi' : H \otimes_A \tilde{H}(n) \to A$$

be the $A$-linear, Galois equivariant perfect pairing induced by $\psi$.

We define the $\mathcal{A}$-bilateral derivation

$$\mathcal{D} : \mathcal{A} \to H^1\big(F, H \otimes_{\mathbf{Z}_l} \tilde{H}(n+1)\big)$$

to be the composition of $\mathcal{D}_0$ with the map induced by $j\otimes j$. We define the $\mathcal{A}$-derivation

$$\partial : \mathcal{A} \to H^1\big(F, H \otimes_A \tilde{H}(n+1)\big)$$

to be the composition of $\mathcal{D}$ with the map on cohomology induced by the surjection $H \otimes_{\mathbf{Z}_l} \tilde{H}(n+1) \twoheadrightarrow H \otimes_A \tilde{H}(n+1)$. We regard $H$ and $\tilde{H}$ as $\mathcal{A}$-modules via the map $\mathcal{A} \to B^* \to A$.

We can use the following diagram to pass from our constructions above to the Galois cohomology of $\operatorname{End}_A H(1)$:

$$
\begin{array}{ccccc}
V \otimes_{\mathbf{Z}_l} V(n) & \xrightarrow{\operatorname{id} \otimes \varphi_r} & V \otimes_{\mathbf{Z}_l} V^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} V \\
\downarrow{\scriptstyle \operatorname{id} \otimes \xi \otimes w} & & \downarrow{\scriptstyle \operatorname{id}} & & \downarrow{\scriptstyle \operatorname{id}} \\
V \otimes_{\mathbf{Z}_l} \tilde{V}(n) & \xrightarrow{\operatorname{id} \otimes \varphi_r'} & V \otimes_{\mathbf{Z}_l} V^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} V \\
\downarrow{\scriptstyle j \otimes j} & & \downarrow{\scriptstyle j \otimes i^\dagger} & & \downarrow{\scriptstyle f \mapsto jfi} \\
H \otimes_{\mathbf{Z}_l} \tilde{H}(n) & \xrightarrow{\operatorname{id} \otimes \psi_r} & H \otimes_{\mathbf{Z}_l} H^\dagger & \longrightarrow & \operatorname{End}_{\mathbf{Z}_l} H \\
\twoheaddownarrow & & \twoheaddownarrow & & \downarrow \\
H \otimes_A \tilde{H}(n) & \xrightarrow{\operatorname{id} \otimes \psi_r'} & H \otimes_A H^\dagger & \longrightarrow & \operatorname{End}_A H
\end{array}
\tag{10.1}
$$

All of these maps are Galois equivariant. The maps are $B^*$-linear except for $\operatorname{id} \otimes \varphi_r$ and $\operatorname{id} \otimes \xi \otimes w$, both of which interchange the action of $B_*$ and $B^*$.

## 11. Untwistings and cycle classes

It will be useful to understand the behavior of certain cycle classes under the top row of (10.1). Let $f : X \to X$ be a morphism and let $\Gamma_f$ be the graph of $f$ in $X \times X$: that is, it is the scheme-theoretic image of the morphism

$$\operatorname{id} \times f : X \to X \times X.$$

By [**FK88**, pp. 155–156] the image of the cycle class

$$s(\Gamma_f) \in H^{2n}\big(X_{\bar{F}} \times X_{\bar{F}}, \mathbf{Z}_l(n)\big)$$

under the maps

$$H^{2n}\big(X_{\bar{F}} \times X_{\bar{F}}, \mathbf{Z}_l(n)\big) \twoheadrightarrow V \otimes_{\mathbf{Z}_l} V(n) \xrightarrow{\operatorname{id} \otimes \varphi_r} V \otimes_{\mathbf{Z}_l} V^\dagger \to \operatorname{End}_{\mathbf{Z}_l} V \tag{11.1}$$

is nothing other than the endomorphism $f^*$ of $V$.

We denote by $\Gamma_f^{\#}$ the scheme-theoretic image of

$$f \times \operatorname{id} : X \to X \times X.$$

Again by [**FK88**, pp. 155–156] the image of $s(\Gamma_f^{\#})$ under (11.1) is the Poincaré adjoint $f^{*\mathrm{adj}}$ of $f^*$. It is characterized by the equality

$$\varphi(f^{*\mathrm{adj}} v, v') = \varphi(v, f^* v')$$

for all $v, v'$.

## 12. Derivations modulo $\eta$

We return now to the notation and hypotheses of Section 10; in particular we assume that we have an untwisting $w$ and a dualizing maximal ideal $\mathfrak{m}$. Fix also a Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$. We have a bilateral derivation

$$\mathcal{D} : \mathcal{A} \to H^1\big(F, H \otimes_{\mathbf{Z}_l} \tilde{H}(n+1)\big)$$

and a derivation

$$\partial : \mathcal{A} \to H^1\big(F, H \otimes_A \tilde{H}(n+1)\big)$$

determined by this collection of data.

Let $I$ be the kernel of the surjection $\mathcal{A} \to A$. By Lemma B.6.1, $\mathcal{D}$ and $\partial$ induce $\mathcal{A}$-module homomorphisms

$$\tilde{\mathcal{D}} : I/I^2 \to H^1\big(F, H \otimes_{\mathbf{Z}_l} \tilde{H}(n+1)\big)_\delta$$
$$\tilde{\partial} : I/I^2 \to H^1\big(F, H \otimes_A \tilde{H}(n+1)\big).$$

By Lemma B.6.2, our choice of Gorenstein trace yields an $A$-linear Galois equivariant isomorphism

$$(H \otimes_{\mathbf{Z}_l} H)_\delta \cong H \otimes_A H$$

fitting into a commutative diagram

$$\begin{array}{ccc} (H \otimes_{\mathbf{Z}_l} H)_\delta & \hookrightarrow & H \otimes_{\mathbf{Z}_l} H \\ \downarrow \simeq & & \downarrow \\ H \otimes_A H & \xrightarrow{\eta} & H \otimes_A H \end{array} \tag{12.1}$$

Here $\eta$ is the congruence element for tr. If we assume that every Jordan-Holder factor of $\tilde{H}(n+1) \otimes_A H$ (as a $G_F$-module) has no $G_F$-invariants, then combining this with Lemma B.6.3 we can view $\tilde{\mathcal{D}}$ as a map

$$\mathcal{D}' : I/I^2 \to H^1\big(F, H \otimes_A \tilde{H}(n+1)\big),$$

which by (12.1) satisfies $\eta\mathcal{D}' = \tilde{\partial}$. The following proposition is an immediate consequence.

PROPOSITION 12.1. *Let $W = H \otimes_A \tilde{H}(n+1)$. There exists an $A$-derivation $\Theta : A \to H^1(F, W/\eta W)$ fitting into a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{A} & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \tilde{\mathcal{D}} & & \downarrow \partial & & \downarrow \Theta & & \\ & & H^1(F,W) & \xrightarrow{\eta} & H^1(F,W) & \longrightarrow & H^1(F, W/\eta W) & & \end{array}$$

PROOF. The bottom exact sequence is part of the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \tilde{H} \otimes_A H \xrightarrow{\eta} \tilde{H} \otimes_A H \longrightarrow \tilde{H} \otimes_A H/\eta \to 0.$$

Note that the first map is injective since $\eta$ is a non-zero divisor by Lemma B.2.2 and the definition of dualizing. The commutativity of the first square is the relationship $\eta\tilde{\mathcal{D}} = \tilde{\partial}$, and the map $\Theta$ is the induced map on cokernels. □

Of course, we can use the identifications of (10.1) to regard $\Theta$ as a derivation

$$\Theta : A \to H^1\big(F, \operatorname{End}_A H/\eta H(1)\big).$$

CHAPTER 9

# Construction of geometric Euler systems

In this chapter we combine the results of Chapters VII and VIII to give geometric conditions for the existence of geometric Euler systems.

## 1. Divisorial liftings of cycles

Let $F$ be a global field and let $S$ be an open subscheme of the spectrum of the ring of integers of $F$. Let $\mathfrak{X}$ be a smooth proper $S$-scheme of relative dimension $n$ and let $X$ be the generic fiber of $\mathfrak{X}$. For $v$ a closed point of $S$, we will often need to consider liftings of cycles on the special fiber $\mathfrak{X}_{k_v}$ up to $\mathfrak{X}$. The relevant notion of lifting is the following.

DEFINITION 1.1. Let $Z$ be a codimension $m$ cycle on $\mathfrak{X}_{k_v}$. We say that a finite set $\{(Z_i, f_i)\}$ of pairs of codimension $m$ cycles $Z_i$ on $X$ and rational functions $f_i$ on $Z_i$ is a *divisorial lifting of* $Z$ if

$$\sum \operatorname{div}_{\mathfrak{Z}_i} f = Z;$$

here $\mathfrak{Z}_i$ is the closure of $Z_i$ in $\mathfrak{X}$ and $Z$ is considered as a vertical cycle on $\mathfrak{X}$.

The first step in constructing a divisorial lifting of a cycle $Z$ is to find a cycle $Z'$ on $X$ which has $Z$ as an irreducible component over $k_v$. This can be done by using a complete intersection containing $Z$ as an irreducible component; any complete intersection can easily be lifted to $X$. It is much harder to find a rational function with trivial divisor on $Z'$ which separates out $Z$ over $k_v$. (Although if $Z$ itself is a complete intersection one can use the methods of Lemma 1.2 for this.)

Note that by definition a divisorial lifting $\sum(Z_i, f_i)$ of $Z$ in $\mathfrak{X}_{k_v}$ has no divisor on any fibers of $\mathfrak{X} \to S$ other than the fiber over $v$. In particular, it has no divisor on $X$, so that $\sum(Z_i, f_i)$ defines an element of $E_2^{m,-m-1}(X)$ by (VI.5.3). Thus a divisorial lifting of $Z$ yields an element $\sum(Z_i, f_i)$ of $E_2^{m,-m-1}(X)$ such that

$$\operatorname{div}_w\big(\sum(Z_i, f_i)\big) = \begin{cases} 0 & w \neq v; \\ Z & w = v. \end{cases}$$

Here $\operatorname{div}_w$ is the map denoted $\operatorname{div}_{k_w}$ in Section VII.1.

The simplest example of divisorial liftings are given by the following lemma.

LEMMA 1.2. *Let $Z$ be a codimension $m$ cycle on $X$ with closure $\mathfrak{Z}$ on $\mathfrak{X}$. Let $v$ be a closed point of $S$ and let $\mathfrak{p}$ be the corresponding prime of $\mathcal{O}_F$. Let $\mathfrak{Z}_v$ be the special fiber of $\mathfrak{Z}$ at $v$. Then $h\mathfrak{Z}_v$ admits a divisorial lifting to $X$, where $h$ is the order of $\mathfrak{p}$ in the ideal class group of $\mathcal{O}_{F,S}$.*

PROOF. By definition of the ideal class group, the ideal $\mathfrak{p}^h$ is principal; let $\pi$ be a generator. Then $\pi$ is a regular function on $S$, non-vanishing away from $v$ and vanishing to order $h$ at $v$. The pair $(Z, \pi)$ is thus a divisorial lifting of $h\mathfrak{Z}_v$. □

Of course, it is unreasonable to expect all cycles on special fibers of $\mathfrak{X}$ to admit liftings as in Lemma 1.2; this is why we introduced the more general notion of divisorial liftings.

Divisorial liftings are designed to give useful elements for Theorem VII.1.1. We will also need to consider the more subtle conditions at certain bad places. Assume for this that $F$ is a number field. Fix a prime $l$ and set $V = H^{2m}(X_{\bar{F}}, \mathbf{Z}_l(m+1))$. Let $T$ be a torsion-free $l$-adic $G_F$-module over $\mathbf{Z}_l$ equipped with a map $V \to T$. We will consider the composition of the Flach map with the projection $V \to T$:

$$\sigma : E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \to H^1(F,T).$$

Definition 1.3. Let $K$ be the completion of $F$ at a place $v$ above $l$. An element $\sum(Z_i, f_i)$ of $E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l}$ is said to be *cohomologically deRham* (resp. *cohomologically crystalline*) *at* $v$ *for* $T$ if $\sigma(\sum(Z_i, f_i))$ lies in $H^1_f(K,T)$ for the deRham (resp. crystalline) local finite/singular structure on $T$.

We have the following sufficient conditions for a pair $(Z, f)$ to be cohomologically deRham or crystalline. Recall that we have a cycle class map

$$s : A^m X \to V(-1) \to T(-1).$$

Lemma 1.4. *Let $K$ be the completion of $F$ at a place $v$ above $l$ and let $\{Z_i\}$ be codimension $m$ cycles on $X$. If $s(Z_i)$ vanishes for each $i$, then each element of $E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l}$ of the form $\sum(Z_i, f_i)$ (for any rational functions $f_i$ on the $Z_i$) is cohomologically deRham.*

*Let $R$ denote the ring of integers of $K$ and let $\mathfrak{Z}_i$ denote the closure of $Z_i$ in $\mathfrak{X}_R$. If it is further possible to realize each $\mathfrak{X}_R - \mathfrak{Z}_i$ as the complement of a normal crossings divisor in a smooth proper variety over $R$ (for example, by embedded resolution of singularities of the $\mathfrak{Z}_i$), then each $\sum(Z_i, f_i)$ is cohomologically crystalline as well.*

Proof. This follows immediately from Proposition VII.10.1 and the definitions of cohomologically deRham and crystalline. □

## 2. Construction of partial Euler systems

In this section we describe the geometric data required to use the Flach map to construct partial geometric Euler systems. Let $F$, $S$ and $\mathfrak{X}$ be as before; we once again allow $F$ to have positive characteristic. Fix an integer $m$ and a prime $l$ and let $V$ denote $H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))$. (If $F$ is a function field we assume that $l \geq n+2$; we will need this in order to invoke Theorem VII.9.1.) Fix a $\mathbf{Z}_l$-algebra $A$ of scalars and let $T$ be a torsion-free $l$-adic $G_F$-module over $A$ equipped with a map $V \to T$ such that the image of $V$ has finite index in $T$. $V$ itself need not have any structure of $A$-module; we do however assume that it is torsion-free, so that by Lemma VI.6.2 we have a Flach map

$$\sigma_m : E_2^{m,-m-1}(X) \to H^1(F,V).$$

We let

$$\sigma : E_2^{m,-m-1}(X) \to H^1(F,T)$$

denote the composition of $\sigma_m$ with the map on cohomology induced by $V \to T$.

$V$ is unramified at all places of $S - \Sigma_l$ by smooth base change. Since the image of $V$ has finite index in $T$ and $T$ is torsion-free, it follows that $T$ is also unramified at all places of $S-\Sigma_l$. We let $V$ and $T$ have the unramified finite/singular structure at all of these places. (We will worry about the structures at the other places later.)

We wish to consider the cycle class map

$$s : A^m\mathfrak{X}_{k_v} \to H^{2m}\big(\mathfrak{X}_{\bar{k}_v}, \mathbf{Z}_l(m)\big)^{G_{k_v}} \cong V(-1)^{G_{k_v}} \to T(-1)^{G_{k_v}}. \tag{2.1}$$

DEFINITION 2.1. Let $v$ be a closed point of $S-\Sigma_l$ and let $\eta$ be an element of $A$. We will say that a collection of codimension $m$ cycles $Z_1, \dots, Z_r$ on $\mathfrak{X}_{k_v}$ *generate $T$ with depth $\eta$* if the $A$-submodule of $T(-1)^{G_{k_v}}$ generated by the $s(Z_i)$ contains $\eta T(-1)^{G_{k_v}}$.

LEMMA 2.2. *Let $Z_1, \dots, Z_r$ be cycles on $\mathfrak{X}_{k_v}$ which generate $T$ with depth $\eta$. Assume that each of the $Z_i$ admit divisorial liftings to $X$. Then there is an $A$-submodule $C$ of $H^1(F,T)$ such that $C_{w,s} = 0$ for $w \in S-\Sigma_l$ distinct from $v$ and such that $C_{v,s}$ has depth $\eta$ in $H^1_s(F_v,T)$.*

PROOF. Let $\sum(Z_{ij}, f_j)$ be a divisorial lifting of $Z_i$ and define $C$ to be the $A$-submodule of $H^1(F,T)$ generated by the $\sigma(\sum(Z_{ij}, f_j))$ for all $i,j$. We will check that this $C$ satisfies the conditions of the lemma.

Let $w$ be a closed point of $S-\Sigma_l$. If $w \neq v$, then the divisor of $\sum(Z_{ij}, f_j)$ vanishes on $\mathfrak{X}_{k_w}$ by the definition of a divisorial lifting; thus by Theorem VII.9.1 $\sigma(\sum(Z_{ij}, f_j))$ vanishes in $H^1_s(F_w, V)$. It therefore vanishes in $H^1_s(F_w,T)$ as well, which shows that $C_{w,s} = 0$.

Applying Theorem VII.9.1 at $v$ shows that $C_{v,s} \subseteq H^1_s(F_v,T)$ is generated by the $s(Z_i)$, where we have identified $H^1_s(F_v,T)$ and $T(-1)^{G_{k_v}}$. Since the $s(Z_i)$ are assumed to fill up $\eta T(-1)^{G_{k_v}}$, we see that $C_{v,s}$ does indeed have depth $\eta$ in $H^1_s(F_v,T)$, as claimed. □

We will consider three different choices of finite/singular structure on $T$. Let $\mathcal{S}_w$ (resp. $\mathcal{S}_d$, resp. $\mathcal{S}_c$) denote the finite/singular structure on $T$ which is weak away from $S$, unramified at $S-\Sigma_l$ and weak (resp. deRham, resp. crystalline) at $\Sigma_l \cap S$.

THEOREM 2.3. *Let $\mathcal{L}$ be a set of closed points of $S-\Sigma_l$. Assume that for each $v \in \mathcal{L}$ there is a set of codimension $m$ cycles of $\mathfrak{X}_{k_v}$ which generate $T$ with depth $\eta$ and which admit divisorial liftings to $X$. Then there is a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth $\eta$ for $T$ with the structure $\mathcal{S}_w$. If further the divisorial liftings are all cohomologically deRham (resp. cohomologically crystalline) then this is a partial Euler system for the structure $\mathcal{S}_d$ (resp. $\mathcal{S}_c$) as well.*

PROOF. This is immediate from Lemma 2.2 and the definitions of cohomologically deRham and crystalline. □

One can combine Theorem 2.3 with Corollary III.3.2 to obtain annihilation results for the Selmer groups of $T^*$; we do not give a precise statement as it becomes notationally quite unpleasant.

## 3. Partial Euler systems on products

In this section we give the simplest method for the construction of geometric Euler systems for $l$-adic $G_F$-modules of endomorphisms. Let $F$, $S$ and $\mathfrak{X}$ be as before. Fix a prime $l$ such that $X$ is cohomologically torsion-free at $l$ (and such that $l \geq n+2$ if $F$ is a function field) and some $m \leq n$. Set $V = H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))$. Let $d$ denote the rank of $V$ as a $\mathbf{Z}_l$-module. Let $T$ be the $l$-adic $G_F$-module $\operatorname{End}^0_{\mathbf{Z}_l} V(1)$ over $\mathbf{Z}_l$. Let $\mathcal{S}_w$, $\mathcal{S}_d$ and $\mathcal{S}_c$ denote the finite/singular structures on $T$ analogous to those in the previous section; as in Chapter IV, control of the Selmer groups of $T^*$ has implications for the deformation theory of $V$.

As in Section VIII.9, our assumption on the cohomology of $X$ yields a canonical map

$$H^{2n}\big(X_{F_s} \times X_{F_s}, \mathbf{Z}_l(n+1)\big) \to V \otimes_{\mathbf{Z}_l} V(n+1) \cong \operatorname{End}_{\mathbf{Z}_l} V(1) \twoheadrightarrow T \tag{3.1}$$

via the Künneth projection and Poincaré duality. We will produce an Euler system for $T$ from the geometry of $X_{F_s} \times X_{F_s}$. We first need to specify at which places to form our Euler system.

Definition 3.1. A $d \times d$ matrix $\tau$ over $\mathbf{F}_l$ is said to be *of general type* if:

- The characteristic polynomial and the minimal polynomial of $\tau$ coincide;
- $\dim_{\mathbf{F}_l}\{M \in M_n\mathbf{F}_l \mid M\tau = \tau M\} = d$.

One checks easily that $\tau$ is of general type if $\tau$ has distinct eigenvalues.

We write $\Gamma_{v,i}$ for the graph in $\mathfrak{X}_{k_v} \times \mathfrak{X}_{k_v}$ of the $i^{\text{th}}$ power of Frobenius on $\mathfrak{X}_{k_v}$; $\Gamma_{v,0}$ is nothing other than the diagonal correspondence.

Lemma 3.2. *Let $\tau$ be a matrix of general type. Let $v$ be a place of $F$ such that $\operatorname{Fr}(v)$ acts on $V/lV \cong \mathbf{F}_l^d$ as a conjugate of $\tau$. Then the cycles $\Gamma_{v,1}, \ldots, \Gamma_{v,d-1}$ generate $T$ via (3.1).*

Proof. Fix an identification of $\operatorname{End}_{\mathbf{F}_l} V/lV$ with $M_n\mathbf{F}_l$ such that $\operatorname{Fr}(v)$ corresponds to $\tau$. Then $(\operatorname{End}_{\mathbf{F}_l} V/lV)^{G_{k_v}}$ identifies with the set of elements of $M_n\mathbf{F}_l$ which commute with $\tau$. Since $\tau$ is of general type, the matrices $\mathrm{id}, \tau, \tau^2, \ldots, \tau^{d-1}$ generate (as an $\mathbf{F}_l$-vector space) the subspace of $M_n\mathbf{F}_l$ of matrices which commute with $\tau$. We conclude that $(\operatorname{End}_{\mathbf{F}_l} V/lV)^{G_{k_v}}$ is generated (as $\mathbf{F}_l$-vector space) by

$$\mathrm{id}, \operatorname{Fr}(v), \operatorname{Fr}(v)^2, \ldots, \operatorname{Fr}(v)^{d-1}.$$

The identity matrix corresponds to the scalars, so $\operatorname{Fr}(v), \ldots, \operatorname{Fr}(v)^{d-1}$ generate $(\operatorname{End}^0_{\mathbf{F}_l} V/lV)^{G_{k_v}}$. By Nakayama's lemma we conclude that $\operatorname{Fr}(v), \ldots, \operatorname{Fr}(v)^{d-1}$ generate $(\operatorname{End}^0_{\mathbf{Z}_l} V)^{G_{k_v}}$, which is equivalent to the statement of the lemma since the cycle class of $\Gamma_{v,i}$ in $\operatorname{End}^0_{\mathbf{Z}_l} V$ is just $\operatorname{Fr}(v)^i$ by Section VIII.11 and [**FK88**, Chapter II, Section 4]. □

Of course, if $\tau$ is not in the image of $G_F \to \operatorname{End}_{\mathbf{Z}_l} V$, then there are no places as in Lemma 3.2.

Theorem 3.3. *Let $\tau$ be a $d \times d$ matrix over $\mathbf{F}_l$ of general type and let $\mathcal{L}$ denote the set of places of $S - \Sigma_l$ with Frobenius conjugate to $\tau$ on $V/lV$. Suppose that there is an integer $\eta$ such that for each $v \in \mathcal{L}$, the cycles $\eta\Gamma_{v,1}, \ldots, \eta\Gamma_{v,d-1}$ admit divisorial liftings to $X$. Then there is a partial Euler system $\{C^v\}_{v \in \mathcal{L}}$ for $T$ of depth $\eta$ with the structure $\mathcal{S}_w$. If these liftings are also cohomologically deRham (resp. cohomologically crystalline) then the Euler system is for $\mathcal{S}_d$ (resp. $\mathcal{S}_c$) as well.*

Proof. This follows immediately from Lemma 3.2 and Theorem 2.3. □

## 4. Construction of Flach systems in the self-adjoint case

In this section we will refine the results of the previous section, via the methods of Sections VIII.8 and VIII.9, to produce Flach systems. Let $F$ be a number field with at least one real embedding and such that $F_v$ is absolutely unramified for every $v \in \Sigma_l$. Let $S$ be an open subscheme of the spectrum of the ring of integers of $F$. Let $\mathfrak{X}$ be a smooth proper $S$-scheme of relative dimension $n$ with generic fiber $X$.

We assume that $n$ is odd. (This assumption is necessary to insure that complex conjugation will act as a non-scalar; to consider the case of even dimension one needs to use the methods of the non-self-adjoint case.) Fix a choice $\tau$ of complex conjugation for $F$.

Fix a prime $l$ such that $S$ contains the set $\Sigma_l$ of places of $F$ above $l$. Set $V = H^n(X_{\bar{F}}, \mathbf{Z}_l)$. We assume that $X$ is cohomologically torsion-free at $l$.

Let $\mathcal{A}$ be a commutative $l$-adic algebra of correspondences on $X$. We assume for this section that $\mathcal{A}$ is self-adjoint, and we let $B$ denote the image of $\mathcal{A}$ in $\operatorname{End}_{\mathbf{Z}_l} V$. Assume also that we have a dualizing maximal ideal $\mathfrak{m}$ of $B$; set $A = B_{\mathfrak{m}}$ and $H = V \otimes_B A$. Set $T = \operatorname{End}^0_A H(1)$. We consider $H$ and $T$ as $l$-adic $G_F$-modules over $A$.

Let $k$ denote the residue field of $A$. Since $\mathfrak{m}$ is dualizing, $A$ is Gorenstein and $H$ is free of rank 2 over $A$. We fix a Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$; let $\eta \in A$ be the associated congruence element.

We need to check that $H$ is a Galois representation of Taylor-Wiles type; we also need to check the conditions of Section IV.4 required to discuss the existence of a Flach system. Note that $H$ is unramified away from the set $\Sigma$ consisting of $\Sigma_l$ and the places of $F$ not in $S$. $H$ is also crystalline at every place of $v$ by [**Fal89**] and [**FM87**]. The pairing required in the definition is simply the pairing $\psi$ of Section VIII.9. It follows that the determinant of $H$ is $\varepsilon^{-n}$. Since $n$ is odd, this is indeed an odd character. We assume also the following conditions:

(1) For every $v \in \Sigma - \Sigma_l$, $H \otimes_A k$ is minimally ramified at $v$ and the minimally ramified structure at $v$ agrees with the weak structure;
(2) $H \otimes_A k$ and $T \otimes_A k$ are absolutely irreducible over $k$;
(3) $H^1(F(T^*[\mathfrak{a}])/F, T^*[\mathfrak{a}]) = 0$ for every ideal $\mathfrak{a}$ of finite index in $A$;
(4) $A$ is generated by the Hecke operators $T_v$ for $v \notin \Sigma_l$;
(5) $H$ is crystalline of weight $k > l$ for each $v \in \Sigma_l$ for every $v \in \Sigma_l$.

Recall that $T_v$ is defined as the trace of $\operatorname{Fr}(v)$ acting on $H$. Note that by Lemma I.5.2 the first assumption is satisfied in the case of ordinary representations. Let $\mathcal{S}$ denote the finite/singular structure on $T$ which is minimally ramified away from $\Sigma_l$ and crystalline at $\Sigma_l$.

For a place $v \in S - \Sigma_l$, let $\Gamma_v$ denote the graph of Fr on $\mathfrak{X}_{k_v}$. Let $\Gamma_v^{\#}$ denote its transpose. Let $\mathcal{L} = \mathcal{L}_\tau$ denote the set of non-archimedean places of $F$ which have Frobenius conjugate to $\tau$ on $H \otimes_A k$.

LEMMA 4.1. *Fix a place $v \in \mathcal{L}$ and let $a, b$ be integers such that $l$ does not divide $a - b$. Then $a\Gamma_v + b\Gamma_v^{\#}$ generates $T$ with depth $\eta$ (via the cycle class map (3.1)).*

PROOF. Recall that by Lemma IV.3.2, $T(-1)^{G_{k_v}} = (\operatorname{End}^0_A H)^{G_{k_v}}$ is a free rank one $A$-module which is generated by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We must compute the image of the cycle class of $a\Gamma_v + b\Gamma_v^{\#}$ under the map

$$H^{2n}\big(X_{\bar{k}_v} \times X_{\bar{k}_v}, \mathbf{Z}_l(n)\big) \twoheadrightarrow V \otimes_{\mathbf{Z}_l} V(n) \cong \operatorname{End}_{\mathbf{Z}_l} V \twoheadrightarrow \operatorname{End}_{\mathbf{Z}_l} H \to \operatorname{End}_A H,$$

of (VIII.9.1) and (VIII.9.2). The discussion of Section VIII.11 shows that the image of $s(\Gamma_v)$ in $\operatorname{End}_{\mathbf{Z}_l} V$ is the morphism $\operatorname{Fr}^*$ of $V$; here $\operatorname{Fr} : X_{\bar{k}_v} \to X_{\bar{k}_v}$ is the base change of the Frobenius morphism of $X_{k_v}$. By [**FK88**, Chapter II, Section 4], $\operatorname{Fr}^*$ is nothing other than the geometric Frobenius automorphism $\operatorname{Fr}(v)$ of $V$. By (VIII.9.1) this maps to the Frobenius automorphism of $H$ in $\operatorname{End}_{\mathbf{Z}_l} H$. Since $\operatorname{Fr}(v)$

is $A$-linear, (VIII.9.2) and Lemma B.3.3 show that this finally maps to $\eta$ times the Frobenius morphism in $\operatorname{End}_A H$:

$$s(\Gamma_v) \mapsto \eta \operatorname{Fr}(v) \in \operatorname{End}_A H. \tag{4.1}$$

Again by Section VIII.11, $\Gamma_v^{\#}$ maps to the Poincaré adjoint $\operatorname{Fr}(v)^{\mathrm{adj}}$ of Frobenius on $V$. Viewing the Poincaré pairing $\varphi$ as a $\mathbf{Z}_l$-linear, Galois equivariant pairing $V \otimes_{\mathbf{Z}_l} V \to \mathbf{Z}_l(-n)$, we can compute this as follows:

$$\begin{aligned}\varphi\big(v, \operatorname{Fr}(v)v'\big) &= \operatorname{Fr}(v)\varphi\big(\operatorname{Fr}(v)^{-1}v, v'\big)\\ &= \varepsilon(v)^{-n}\varphi\big(\operatorname{Fr}(v)^{-1}v, v'\big)\\ &= \varphi\big(\varepsilon(v)^{-n}\operatorname{Fr}(v)^{-1}v, v'\big).\end{aligned}$$

Thus $\operatorname{Fr}(v)^{\mathrm{adj}} = \varepsilon(v)^{-n}\operatorname{Fr}(v)^{-1}$. The same analysis as for (4.1) now shows that:

$$s(\Gamma_v^{\#}) \mapsto \eta\varepsilon(v)^{-n}\operatorname{Fr}(v)^{-1} \in \operatorname{End}_A H. \tag{4.2}$$

By Lemma IV.3.1 we can choose a basis of $H$ with respect to which $\operatorname{Fr}(v)$ is given by a matrix $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$. Since $H$ has determinant $\varepsilon^{-n}$, we have $\alpha\beta = \varepsilon(v)^{-n}$. Thus $\operatorname{Fr}(v)^{\mathrm{adj}}$ is given by the matrix $\left(\begin{smallmatrix}\beta & 0\\ 0 & \alpha\end{smallmatrix}\right)$. Thus $a\operatorname{Fr}(v) + b\operatorname{Fr}(v)^{\#}$ is just

$$\begin{pmatrix} a\alpha + b\beta & 0\\ 0 & b\alpha + a\beta\end{pmatrix}.$$

This projects to

$$\frac{1}{2}(a-b)(\alpha-\beta)\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \tag{4.3}$$

in $\operatorname{End}_A^0 H$.

We conclude by (4.1) and (4.2) that the image of the cycle class of $a\Gamma_v + b\Gamma_v^{\#}$ in $\operatorname{End}_A^0 H$ is $\eta$ times (4.3). As in the proof of Lemma IV.3.2, $\alpha - \beta$ is a unit in $A$. Since we assumed that $l$ does not divide $a - b$, this is indeed of depth $\eta$ in $(\operatorname{End}_A^0 H)^{G_{k_v}}$, as required. □

THEOREM 4.2. *Assume that for every $v \in \mathcal{L}$ there are integers $a_v, b_v$ such that $l$ does not divide $a_v - b_v$ and such that the cycle $a_v\Gamma_v + b_v\Gamma_v^{\#}$ admits a divisorial lifting to $X \times X$. Assume also that these divisorial liftings are cohomologically crystalline. Then $T$ admits a Flach system of depth $\eta$ for the structure $\mathcal{S}$.*

PROOF. Let $c^v \in H^1(F,T)$ denote the image of the divisorial lifting of $a_v\Gamma_v + b_v\Gamma_v^{\#}$ under the Flach map

$$\sigma : E_2^{n,-n-1}(X) \to H^1(F,T). \tag{4.4}$$

We will show that $\{c^v\}_{v\in\mathcal{L}}$ is a Flach system of depth $\eta$. To do this we must check that the $A$-submodule $C^v$ of $H^1(F,T)$ generated by $c^v$ maps to 0 in $H_s^1(F_w,T)$ for $w \neq v$ and has strict depth $\eta$ at $v$. The conditions for $w \notin \Sigma$ and $v$ are dealt with as in the proof of Lemma 2.2. For $w \in \Sigma - \Sigma_l$, $H_s^1(F_w,T) = 0$ by assumption, so the local condition for $C^v$ is automatic. Finally, the conditions for $w \in \Sigma_l$ are part of the hypotheses. □

## 5. Construction of Flach systems in the general case

We now give the analogue of Theorem 4.2 without the self-adjoint hypothesis. Let $F$, $S$ and $\mathfrak{X}$ be as before. We no longer assume that the dimension $n$ is odd. We again fix a prime $l$ such that $S$ contains $\Sigma_l$ and such that $X$ is cohomologically torsion-free at $l$. Set $V = H^n(X_{\bar{F}}, \mathbf{Z}_l)$. We could also allow $V$ to be a direct summand of $H^n(X_{\bar{F}}, \mathbf{Z}_l)$ as discussed in Section VIII.10.

Let $\mathcal{A}$ be a commutative $l$-adic algebra of correspondences on $X$. Let $B_*$ and $B^*$ denote the images of $\mathcal{A}$ in $\operatorname{End}_{\mathbf{Z}_l} V$ as in Section VIII.10. Assume also that we have an untwisting $(w, \tilde{B}, \xi)$. Let $\mathfrak{m}$ be a dualizing maximal ideal of $B^*$; set $A = B^*_{\mathfrak{m}}$ and $H = V \otimes_{B^*} A$. Since $\mathfrak{m}$ is dualizing, $A$ is Gorenstein; let tr be a fixed choice of Gorenstein trace with associated congruence element $\eta$. Let $\chi : G_F \to A^\times$ denote the determinant character of $H$. We assume that $\chi$ is odd.

We again check that $H$ is of Taylor-Wiles type and satisfies the assumptions of Section IV.4. As before, $H$ is unramified away from the set $\Sigma$ consisting of $\Sigma_l$ and the places of $F$ not in $S$, and $H$ is crystalline at every place of $\Sigma_l$. We assume also the conditions 1,2,3,4,5 on $H$ given in Section 4. All other hypotheses are satisfied as before. Let $\mathcal{S}$ denote the finite/singular structure on $T$ which is minimally ramified away from $\Sigma_l$ and crystalline at $\Sigma_l$.

We will need one last piece of data.

Definition 5.1. Let $v$ be a place of $S$. By a *diamond operator* for $v$ we mean an automorphism $\langle v \rangle$ of $X$ such that $j \langle v \rangle^* i = \chi(v)\varepsilon(v)^n$ as an automorphism of $H$.

Let $v$ be a place in $S - \Sigma_l$. Let $\Gamma_v$ denote the graph of $\operatorname{Fr}(v)$ on $\mathfrak{X}_{k_v}$. Assume that there exists a diamond operator $\langle v \rangle$ for $v$ and let $\Gamma'_v$ denote the image of

$$\operatorname{Fr}(v) \times \langle v \rangle : X \to X \times X.$$

Let $\mathcal{L} = \mathcal{L}_\tau$ denote the set of places of $F$ which have Frobenius conjugate to $\tau$ on $H \otimes_A k$.

Lemma 5.2. *Fix a place $v \in \mathcal{L}$ and let $a, b$ be integers such that $l$ does not divide $a - b$. Then $a\Gamma_v + b\Gamma'_v$ generates $T$ with depth $\eta$.*

Proof. This proof is quite close to that of Lemma 4.1. The only difference is the computation of the cycle classes. Applying the analysis of Section VIII.12, we see that the cycle class of $\Gamma_v$ in $\operatorname{End}_A H$ is still just $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, where we have chosen a basis for $H$ as before. The cycle class of $\Gamma'_v$ in $\operatorname{End}_{\mathbf{Z}_l} V$ is $\langle v \rangle^* \operatorname{Fr}(v)^{\mathrm{adj}}$. The Poincaré adjoint of $\operatorname{Fr}(v)$ is still $\varepsilon(v)^{-n} \operatorname{Fr}(v)^{-1}$. We now see from the definition of diamond operators that $\langle v \rangle^* \operatorname{Fr}(v)^{\mathrm{adj}}$ is $\chi(v) \operatorname{Fr}(v)^{-1}$. Since $\chi(v)$ is the determinant of $\operatorname{Fr}(v)$ on $H$, from here the analysis is exactly as in Lemma 4.1. □

Theorem 5.3. *Suppose that for every $v \in \mathcal{L}$ there are integers $a_v, b_v$ such that $l$ does not divide $a_v - b_v$ and such that the cycle $a_v\Gamma_v + b_v\Gamma'_v$ admits a divisorial lifting to $X \times X$. Assume also that these divisorial liftings are cohomologically crystalline). Then $T$ admits a Flach system of depth $\eta$ for the structure $\mathcal{S}$.*

Proof. This is proven in the same way as Theorem 4.2, using Lemma 5.2 instead of Lemma 4.1. □

## 6. Construction of cohesive Flach systems

It is quite easy to extend the methods of the previous sections to construct cohesive Flach systems. We continue with the hypotheses of the previous section; we will no longer treat the self-adjoint case separately. (Of course, the self-adjoint case is a special case of the general case via the untwisting $(\mathrm{id}, B^*, 1)$. Note also that in the self-adjoint case diamond operators are given simply by the identity map.) If $\omega$ is a marking on the curve $X$ (that is, a rational section of some invertible exterior power of a sheaf of differentials on $X$), then we write $f_\alpha$ for the induced rational function on a correspondence $\alpha$ as in Section VIII.4. Note that if $(\alpha, f_\alpha)$ is a divisorial lifting (of anything) only if $\alpha$ is admissible for the marking $\omega$.

THEOREM 6.1. *Let $\omega$ be an admissible marking on $X$. Assume that for all $v \notin \Sigma_l$ there is a correspondence $\mathfrak{T}_v \in \mathcal{A}$ such that $(\mathfrak{T}_v, f_{\mathfrak{T}_v})$ is a divisorial lifting of $a_v\Gamma_v + b_v\Gamma_v^{\#}$; here $a_v, b_v$ are integers such that $l$ does not divide $a_v - b_v$. Assume also that $\mathfrak{T}_v^*$ yields the Hecke operator $T_v$ in $A$. Further assume that the $(\mathfrak{T}_v, f_{\mathfrak{T}_v})$ are cohomologically crystalline. Then $T$ admits a cohesive Flach system of depth $\eta$ for the structure $\mathcal{S}$. If the differences $a_v - b_v$ are a constant independent of $v$, then the cohesive Flach system is of Eichler-Shimura type of weight twice this constant.*

PROOF. The classes $c^v$ are defined to be $\sigma(\mathfrak{T}_v, f_{\mathfrak{T}_v})$, with $\sigma$ the general case of (4.4). The local analysis is as in the previous constructions; note that the fact that $c^v$ maps to 0 in $H^1_s(F_w, T/\eta T)$ for all $v, w$ is immediate from the fact that the map

$$\mathrm{End}_{\mathbf{Z}_l} H \to \mathrm{End}_A H$$

is multiplication by $\eta$ on $A$-linear maps. The derivation $\Theta : A \to H^1(F, T/\eta T)$ is that constructed in Proposition VIII.12.1. This completes the construction of the cohesive Flach system.

The fact that the cohesive Flach system is of Eichler-Shimura type if the differences are constant follows immediately from the definition of Eichler-Shimura type and the fact that $\mathrm{Ver}(v)$ (as defined in Section IV.6) agrees with the cycle class of $\Gamma_v'$ as computed in the proof of Lemma 5.2. □

# Part 3

# Examples

CHAPTER 10

# The modular curve $X_0(N)$

In this chapter and the next we construct an explicit cohesive Flach system of Eichler-Shimura type for representations associated to weight 2 newforms with trivial character.

## 1. The geometry of $X_0(N)$

We begin by recalling the basic geometry of the modular curve $X_0(N)$. We will work logically somewhat out of order, as we will actually define $X_0(N)$ in terms of $X_1(N)$ in the next chapter. We give most references to the summary [**DI95**, Sections 8 and 9], which in turn contains references to the standard sources [**DR73**] and [**KM85**]. See also [**Gro90**, Sections 2 and 3] and [**MW84**, Chapter 2, Sections 3-5].

**1.1. The model $\mathfrak{X}_0(N)$.** Let $E/S$ be a generalized elliptic curve over a $\mathbf{Z}[\frac{1}{N}]$-scheme $S$. We define a $\Gamma_0(N)$*-structure* on $E/S$ to be a finite flat subgroup scheme $C$ with all geometric fibers cyclic of order $N$; we further require that $C$ meet every irreducible component of fibers of $E/S$ which are Néron polygons. In particular, we see that a Néron $d$-gon can only have a $\Gamma_0(N)$-structure if $d$ divides $N$. We consider two $\Gamma_0(N)$-structures $(E/S, C)$ and $(E'/S, C')$ to be isomorphic if there is an $S$-isomorphism $E \xrightarrow{\simeq} E'$ taking $C$ to $C'$.

$\mathfrak{X}_0(N)$ is a $\mathbf{Z}[\frac{1}{N}]$-scheme which coarsely represents the $\Gamma_0(N)$-moduli problem; see [**DI95**, Sections 9.2 and 9.3]. $\mathfrak{X}_0(N)$ is a smooth, proper, geometrically connected $\mathbf{Z}[\frac{1}{N}]$-scheme of relative dimension 1; this will all follow from our description of $\mathfrak{X}_0(N)$ in terms of $\mathfrak{X}_1(N)$ in the next chapter, together with [**KM85**, Theorem 7.1.3]. In fact, $\mathfrak{X}_0(N)$ admits a proper, regular model over $\mathbf{Z}$; see [**DI95**, Section 8.3].

**1.2. The degeneracy maps.** For all $N$ dividing $M$, there is a natural *degeneracy map*

$$j_{M,N} : \mathfrak{X}_0(M) \to \mathfrak{X}_0(N);$$

here we are taking the model of $\mathfrak{X}_0(M)$ over $\mathbf{Z}[\frac{1}{N}]$ obtained from the proper regular model over $\mathbf{Z}$. $j_{M,N}$ is defined on the moduli level by sending the $\Gamma_0(M)$-structure $(E/S, C)$ to the $\Gamma_0(N)$-structure $(E/S, C_M)$, where $C_N$ is the unique subgroup scheme of $C$ of order $N$. We will also need an alternate degeneracy map in the case that $p$ is a prime not dividing $N$:

$$j'_{Np,N} : \mathfrak{X}_0(Np) \to \mathfrak{X}_0(N).$$

On moduli, $j'_{Np,N}$ sends $(E/S, C)$ to the pair $((E/C_p)/S, C/C_p)$ where $C_p$ is the unique subgroup of $C$ of order $p$. Both maps $j_{Np,N}$ and $j'_{Np,N}$ are étale over $\operatorname{Spec} \mathbf{Z}[\frac{1}{Np}]$ away from the cusps (which we will define in the next section); see

[**Gro90**, Section 3]. One should keep in mind that the moduli definitions above become more complicated (including contractions of irreducible components) on Néron polygons.

**1.3. The cusps.** $\mathfrak{X}_0(N)$ has a certain finite set of distinguished horizontal closed subschemes called the *cusps*; in terms of the moduli problem they correspond to Néron polygons. For our purposes it will suffice to describe the cusps over an arbitrary algebraically closed field $k$ of characteristic prime to $N$. (In fact, our description is valid over algebraically closed fields of any characteristic so long as we use models of $\mathfrak{X}_0(N)$ over $\mathbf{Z}$ and we use cyclic in the sense of [**KM85**, Chapter 1, Section 4].) We will say that a cusp of $\mathfrak{X}_0(N)_k$ is of *type* $d$ if the corresponding Néron polygon is a $d$-gon; as we observed above, $d$ must be a divisor of $N$.

To begin we allow $N$ to be arbitrary. Fix an integer $d$ dividing $N$ and let $\mathcal{E}_d = \mathbf{G}_\mathrm{m} \times \mathbf{Z}/d\mathbf{Z}$ denote the Néron $d$-gon over $k$. We will classify $\Gamma_0(N)$-structures on $\mathcal{E}_d$; these are the type $d$ cusps of $\mathfrak{X}_0(N)_k$. Fix a primitive $N^{\text{th}}$ root of unity $\zeta$ in $k$.

The $N$-torsion on $\mathcal{E}_d$ is $\mu_N \times \mathbf{Z}/d\mathbf{Z}$. We will call an element of $\mathcal{E}_d[N]$ *primary* if it has exact order $N$ and projects to $1 \in \mathbf{Z}/d\mathbf{Z}$. Note that by definition every $\Gamma_0(N)$-structure on $\mathcal{E}_d$ is generated by a primary element. Thus to determine the type $d$ cusps it suffices to classify such subgroups up to automorphisms of $\mathcal{E}_d$. One sees immediately that the primary elements of $\mathcal{E}_d[N]$ are of the form $\zeta^a \times 1$ for $a$ relatively prime to $\frac{N}{d}$. In particular, there are $d\phi(\frac{N}{d})$ primary elements; here $\phi$ is the Euler totient function.

Two primary elements $\zeta^a \times 1$ and $\zeta^b \times 1$ generate the same subgroup of $\mathcal{E}_d[N]$ precisely when $a \equiv b \pmod{d}$. We denote this subgroup by $S_{d,a}(N)$; here the second subscript is understood to run through those congruence classes in $\mathbf{Z}/d\mathbf{Z}$ which contain representatives relatively prime to $\frac{N}{d}$. One finds that there are

$$\frac{d\phi\left(\frac{N}{d}\right)\phi(d)}{\phi(N)}$$

such subgroups.

These subgroups may still be related by automorphisms of $\mathcal{E}_d$ and thus give rise to the same $\Gamma_0(N)$-structure. By [**DR73**, Chapter I], the automorphism group of $\mathcal{E}_d$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} \ltimes \mu_d$. The $\mathbf{Z}/2\mathbf{Z}$ acts by "inversion" and thus preserves all of the subgroups $S_{d,a}(N)$. On the other hand, $\xi \in \mu_d$ acts on a primary element $\zeta^a \times 1$ by $\zeta^a \times 1 \mapsto \zeta^a\xi \times 1$. It follows that

$$\bigl(\mathcal{E}_d, S_{d,a}(N)\bigr) \cong \bigl(\mathcal{E}_d, S_{d,b}(N)\bigr)$$

precisely when $a \equiv b \pmod{g}$ with $g = \gcd(d, \frac{N}{d})$. We will write the corresponding cusp of $\mathfrak{X}_0(N)_k$ as $C_{d,a}(N)$ with $a \in \mathbf{Z}/g\mathbf{Z}$; in fact, since the only additional condition is that $a$ is relatively prime to $\frac{N}{d}$, we see that $a$ runs precisely through $(\mathbf{Z}/g\mathbf{Z})^\times$. In particular, there are $\phi(g)$ cusps of type $d$. The absolute ramification degree of $C_{d,a}(N)$ over the unique cusp of $\mathfrak{X}_0(1)_k$ is equal to the number of subgroups of $\mathcal{E}_d$ contained in $C_{d,a}(N)$; this is just $\frac{d}{g}$.

We will also need to understand the behavior of the cusps under the maps $j_{Np,N}$ and $j'_{Np,N}$ for $p$ not dividing $N$. We now restrict to the case when $N$ is squarefree. In this case $\mathfrak{X}_0(N)_k$ has a unique cusp of each type $d$ for dividing $N$; we denote it by $C_d(N)$. Similarly, $\mathfrak{X}_0(Np)_k$ has a unique cusp $C_d(Np)$ for each $d$

dividing $Np$. One now computes easily the image of each $C_d(Np)$ under $j_{Np,N}$ and $j'_{Np,N}$; one finds that

$$j^{-1}C_d(N) = \{C_d(Np), C_{dp}(Np)\}$$

with ramification degrees 1 and $p$, respectively:

(1.1)
$$\begin{array}{ccccc} C_d(Np) & & & & C_{dp}(Np) \\ & \overset{1}{\searrow} & & \overset{p}{\swarrow} & \\ & & C_d(N) & & \end{array}$$

The behavior under $j'_{Np,N}$ is the same except that the cusp of type $d$ is exchanged with the cusp of type $dp$:

(1.2)
$$\begin{array}{ccccc} C_d(Np) & & & & C_{dp}(Np) \\ & \overset{p}{\searrow} & & \overset{1}{\swarrow} & \\ & & C_d(N) & & \end{array}$$

**1.4. The Hecke correspondences.** Fix a prime $p$ not dividing $N$. We define the $p^{th}$ *Hecke correspondence* $\mathfrak{T}_p$ on $\mathfrak{X}_0(N)$ to be the scheme-theoretic image of the map

$$j_{Np,N} \times j'_{Np,N} : \mathfrak{X}_0(Np) \to \mathfrak{X}_0(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_0(N).$$

$\mathfrak{T}_p$ is birational to $\mathfrak{X}_0(Np)$ away from characteristic $p$ and has pure codimension 1 in $\mathfrak{X}_0(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_0(N)$. It is possible to view $\mathfrak{T}_{p,\mathbf{F}_p}$ as an algebraic self-correspondence on $\mathfrak{X}_0(N)_{\mathbf{F}_p}$ (we will explain how in our discussion of the Hecke algebra $\mathbf{T}_0(N)$ below) and we have the *Eichler-Shimura relation*

$$\mathfrak{T}_{p,\mathbf{F}_p} = \Gamma_p + \Gamma_p^{\#}, \qquad (1.3)$$

where $\Gamma_p$ is the graph of the Frobenius morphism on $\mathfrak{X}_0(N)_{\mathbf{F}_p}$ and $\Gamma_p^{\#}$ is its transpose; here we regard $\Gamma_p$ and $\Gamma_p^{\#}$ as algebraic self-correspondences on $X_0(N)_{\mathbf{F}_p}$ in the obvious way. See [**Gro90**, p. 454] and [**DI95**, Section 8.4].

**1.5. The Atkin correspondences.** Fix a prime $p$ dividing $N$. We define a $\Gamma_0(N;p)$-*structure* on a generalized elliptic curve $E/S$ to be a pair $(C, C')$ of finite flat subgroup schemes of $E/S$ of order $N$ and $p$ respectively such that $C \cap C' = 0$. We further require that $C + C'$ meets all irreducible components of fibers of $E/S$ which are Néron polygons. One sees easily that $\Gamma_0(N;p)$-structures exist on Néron $d$-gons only for $d = pd'$ with $d'$ a divisor of $\frac{N}{p}$. We have the obvious notion of an isomorphism of $\Gamma_0(N;p)$-structures. The $\Gamma_0(N;p)$-moduli problem is coarsely represented by a proper, regular $\mathbf{Z}[\frac{1}{N}]$-scheme $\mathfrak{X}_0(N;p)$; see [**MW84**, Chapter 2, Section 5.5].

There are two natural degeneracy maps $j_{N;p,p}$ and $j'_{N;p,p}$ from $\mathfrak{X}_0(N;p)$ to $\mathfrak{X}_0(N)$. The first sends the triple $(E/S, C, C')$ to the pair $(E/S, C)$ and the second sends it to the pair $((E/C')/S, (C+C')/C')$. We define the $p^{th}$ *Atkin correspondence* $\mathfrak{T}_p$ to be the scheme-theoretic image of the map

$$j_{N;p,p} \times j'_{N;p,p} : \mathfrak{X}_0(N;p) \to \mathfrak{X}_0(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_0(N).$$

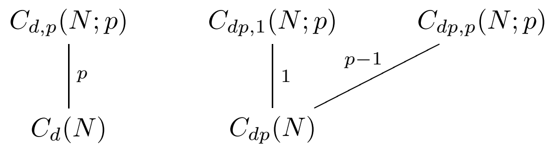
$\mathfrak{T}_p$ is birational to $\mathfrak{X}_0(N;p)$ away from characteristic $p$ and has pure codimension 1 in $\mathfrak{X}_0(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_0(N)$. Often in the literature our $\mathfrak{T}_p$ is denoted $\mathfrak{U}_p$.

We will need to understand how the cusps of $\mathfrak{X}_0(N;p)_k$ sit over the cusps of $\mathfrak{X}_0(N)_k$ for $k$ an algebraically closed field of characteristic not dividing $N$. For simplicity we only give an explicit description in the case that $N$ is squarefree. The analysis of these structures is similar to (although slightly more complicated than) the analysis of Section 1.3.

Let $(\mathcal{E}_{dp}, C, C')$ be a $\Gamma_0(N;p)$-structure. We define the type $(i,j)$ of the corresponding cusps of $\mathfrak{X}_0(N;p)_k$ as follows: we let $i$ (resp. $j$) equal the order of the image of the map $C \to \mathbf{Z}/pd\mathbf{Z}$ (resp. $C' \to \mathbf{Z}/pd\mathbf{Z}$).

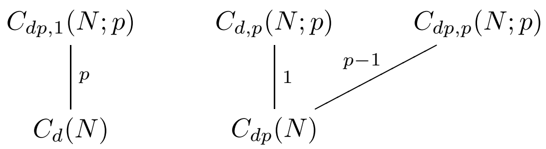
One finds that the cusps of $\mathfrak{X}_0(N;p)_k$ are precisely indexed by these types: the allowable types are $(dp,1)$, $(dp,p)$ and $(d,p)$ for $d$ dividing $\frac{N}{p}$. These are related to the cusps of $\mathfrak{X}_0(N)$ under $j_{N;p,N}$ by:

(1.4)




and under $j'_{N;p,N}$ by

(1.5)




**1.6. The Hecke algebra $\mathbf{T}_0(N)$.** The modular curve $X_0(N)$ is the generic fiber $\mathfrak{X}_0(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \operatorname{Spec} \mathbf{Q}$ of $\mathfrak{X}_0(N)$; it is a smooth projective curve over $\mathbf{Q}$. Write $T_p$ for the Hecke correspondence $\mathfrak{T}_{p,\mathbf{Q}}$ on $X_0(N)$. One can also define Hecke correspondences $T_n$ for all $n$ (using an appropriate moduli interpretation; see [**Roh97**, Chapter 2, Sections 1-3]). Note that the $T_n$ are divisors on $\mathfrak{X}_0(N)$, and therefore are local complete intersections; in particular, they are Cohen-Macaulay. The two projections $T_n \to X_0(N)$ are visibly quasi-finite and proper, so by [**GDb**, Proposition 15.4.2] and [**GDa**, Proposition 4.4.2] they are both finite flat. Checking on the level of geometric points shows that they are both finite, faithfully flat of the same degree. In particular, we see that the Hecke correspondences $T_n$ really are algebraic correspondences in the sense of Definition VIII.1.1. (This argument works over fields of any characteristic relatively prime to $N$ and therefore completes our description of the Eichler-Shimura relation in (1.3).)

The Hecke correspondences satisfy the relations

$$\begin{aligned} T_{mn} &= T_m T_n && \text{for } m, n \text{ relatively prime;} \\ T_{p^n} &= T_{p^{n-1}} T_p - p T_{p^{n-2}} && \text{for } p \text{ not dividing } N; \\ T_{p^n} &= T_p^n && \text{for } p \text{ dividing } N; \end{aligned}$$

see [**Lan76**, Chapter 7, Theorem 2.1]. It follows that the $T_p$ (for all $p$) generate a commutative algebra of correspondences $\mathbf{T}_0(N)$. In fact, $\mathbf{T}_0(N)$ can be generated without $T_l$ for any fixed prime $l$ so long as we assume that $l$ does not divide $N$; see [**DDT97**, Lemma 4.1].

**1.7. Poincaré duality.** Fix a prime $l \geq 7$ and not dividing $N$ and let $V = H^1(X_0(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$; we wish to fit $V$ into the general framework of Section VIII.8. It follows from the formulas in [**MW84**, p. 236] and the fact that the diamond automorphisms act trivially on $V$ that the Hecke operators $T_p$ for $p$ not dividing $N$ are self-adjoint acting on $V$. However, here we must make the unfortunate assumption that all $T_p$ are self-adjoint acting on $V$. This is certainly false in general; we make this assumption so that we can proceed in the simpler self-adjoint case. We will remove it in the next chapter. Given this, the Poincaré pairing $\varphi : V \otimes_{\mathbf{Z}_l} V(1) \to \mathbf{Z}_l$ is a $\mathbf{T}_0(N)$-hermitian, skew-symmetric, Galois equivariant perfect pairing.

**1.8. Non-Eisenstein maximal ideals.** Let $B$ denote the image of $\mathbf{T}_0(N)$ in $\operatorname{End}_{\mathbf{Z}_l} V$; by self-adjointness this is independent of which map we take. We wish to find a dualizing maximal ideal $\mathfrak{m}$ of $B$. There is a very well developed theory of such ideals; see [**DDT97**, Chapter 4] and [**Til97**] for an exposition. The first step is to find a maximal ideal $\mathfrak{m}$ of $B$ associated to a newform and such that the Galois representation $V \otimes_B k$ (where $A = B_{\mathfrak{m}}$ and $k = A/\mathfrak{m}_A$ as always) is absolutely irreducible. This occurs precisely when $\mathfrak{m}$ is not Eisenstein. (Recall that $\mathfrak{m}$ is said to be *Eisenstein* if $T_p \equiv p + 1 \pmod{\mathfrak{m}}$ for all primes $p$ congruent to 1 modulo $N$; see [**DDT97**, Section 4.3, esp. Lemma 4.12].) Since $l$ does not divide $N$, [**Til97**, Theorem 3.4] and the proofs of its corollaries show that a non-Eisenstein $\mathfrak{m}$ containing $l$ is dualizing in our sense. In particular, $A$ is Gorenstein. Fix for the remainder of the chapter one such maximal ideal $\mathfrak{m}$ and a corresponding choice of Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$. Let $\eta \in A$ denote the corresponding congruence element.

**1.9. The finite/singular structure.** Let $T = \operatorname{End}_A^0 H(1)$ considered as a $G_{\mathbf{Q}}$-module. We give $T$ the finite/singular structure as in Section IX.4: that is, it is unramified away from $Nl$, minimally ramified at $N$ and crystalline at $l$. We will assume below that $N$ is squarefree so that $H$ is ordinary at primes dividing $N$; thus the minimally ramified structure will coincide with the weak structure. We will return to this below.

## 2. The modular unit $\Delta$

We assume from now on that $N$ is squarefree. In order to construct a cohesive Flach system for $T$ as in Chapter IX.6, we first must find an admissible marking for the algebra $\mathbf{T}_0(N)$. Following Flach we choose $\Delta$, the unique normalized rational cusp form of weight 12 and level 1. In fact, we will see that this is essentially the only choice, at least if we constrain the divisor to the cusps. (We prefer not to even contemplate non-cuspidal divisors.)

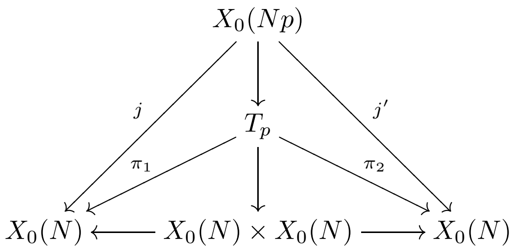
LEMMA 2.1. *The marking $\Delta$ is admissible for the Hecke algebra $\mathbf{T}_0(N)$. Furthermore, the only cusp forms of any weight (and level dividing $N$) which yield admissible markings for $\mathbf{T}_0(N)$ are rational multiples of powers of $\Delta$.*

PROOF. Since one can check if a divisor vanishes after base extension, for this proof all of the varieties and divisors we consider will be over $\bar{\mathbf{Q}}$; we will omit this from our notation. On the modular curve $X(1)$, $\Delta$ lies in $\Omega_{X(1)}^{\otimes 6}$ and has divisor $C_1(1)$. It follows easily that on $X_0(N)$, $\Delta$ lies in $\Omega_{X_0(N)}^{\otimes 6}$ and has divisor

$$\sum_{d|N} dC_d(N). \tag{2.1}$$

(Recall that since $N$ is squarefree its cusps correspond bijectively to the divisors of $N$.)

We must show that each $T_p$ is admissible for the marking $\Delta$. Suppose first that $p$ is a prime not dividing $N$. We have a diagram




To show that $T_p$ is admissible for $\Delta$, we must show that the rational function

$$f_p = \frac{\pi_1^*\Delta}{\pi_2^*\Delta} \in k(T_p)^\times$$

has trivial divisor. We will compute first the divisor of the rational function

$$\frac{j^*\Delta}{j'^*\Delta} \in k\big(X_0(Np)\big)^\times \tag{2.2}$$

on $X_0(Np)$; since the map $X_0(Np) \to T_p$ is birational, we can identify (2.2) with $f_p$.

(2.1) and (1.1) show that the divisor of $j^*\Delta$ on $X_0(Np)$ is

$$\sum_{d|N} dC_d(Np) + dpC_{dp}(Np). \tag{2.3}$$

In the same way, we see from (1.2) that $j'^*\Delta$ has divisor

$$\sum_{d|N} dpC_d(Np) + dC_{dp}(Np). \tag{2.4}$$

Thus on $X_0(Np)$ the rational function $f_p$ has divisor

$$\sum_{d|N} d(1-p)\big(C_d(Np) - C_{dp}(Np)\big). \tag{2.5}$$

Since under $j \times j' : X_0(Np) \to T_n$ both $C_d(Np)$ and $C_{dp}(Np)$ map to $C_d(N) \times C_d(N)$, we see from (2.5) that $f_p$ has trivial divisor on $T_p$. Thus $T_p$ is divisorless for $\Delta$ for all $p$ not dividing $N$.

The same is true for $p$ dividing $N$. We can write the divisor (2.1) on $X_0(N)$ as

$$\sum_{d|\frac{N}{p}} dC_d(N) + dpC_{dp}(N).$$

It follows from (1.4) that on $X_0(N;p)$, $j^*\Delta$ has divisor

$$\sum_{d|\frac{N}{p}} dpC_{d,p}(N;p) + dpC_{dp,1}(N;p) + dp(p-1)C_{dp,p}(N;p).$$

By (1.5), $j'^*\Delta$ has divisor

$$\sum_{d|\frac{N}{p}} dpC_{dp,1}(N;p) + dpC_{d,p}(N;p) + dp(p-1)C_{dp,p}(N;p).$$

Therefore the rational function $f_p$ has trivial divisor on $X_0(N;p)$; it follows that $T_p$ is admissible for the marking $\Delta$ for $p$ dividing $N$ as well.

This leaves the uniqueness assertion. In fact, one checks immediately as above that *any* cusp form is admissible for the sub-algebra of $\mathbf{T}_0(N)$ generated by the $T_p$ for $p$ not dividing $N$. The difficulty arises on the $T_p$ for $p$ dividing $N$. Let $g$ be a rational cusp form of level $N$ and weight $k$, with divisor

$$\sum_{d|N} n_d C_d(N)$$

on $X_0(N)$. Fix a prime $p$ dividing $N$. One checks that the condition that $f_g$ be divisorless for $T_p$ is that

$$n_{pd} = pn_d \tag{2.6}$$

where $d$ runs over the divisors of $\frac{N}{p}$. (This essentially means that $g$ must come from level $\frac{N}{p}$.) The conditions (2.6) for all primes $p$ dividing $N$ show that the $n_d$ are given by

$$n_d = dn_1.$$

Thus the divisor of $g$ is a multiple of the divisor of $\Delta$. Since $X_0(N)$ is a non-singular curve and $\Delta$ is the modular form of weight 1 and minimal positive weight, this implies that $g$ is a multiple of a power of $\Delta$, as claimed. □

## 3. The divisor of $f_p$ in positive characteristic

In this section we prove the following key lemma.

LEMMA 3.1. *For all $p$ not dividing $N$, the pair $(T_p, f_p)$ is a divisorial lifting of* $6(\Gamma_p^{\#} - \Gamma_p)$.

PROOF. Since $f_p$ has no divisor on $T_p$ in characteristic 0, we know that the divisor of $f_p$ on $\mathfrak{T}_p$ has no horizontal component. We must show that it also has no vertical component in characteristics different from $p$ (and not dividing $N$) and that it has the appropriate divisor in characteristic $p$.

Assume first that $r$ is a prime not dividing $Np$. In this case the calculation is exactly the same as the calculation in characteristic 0. Indeed, the degeneracy maps

$$j_{Np,N} : X_0(Np)_{\bar{\mathbf{F}}_r} \to X_0(N)_{\bar{\mathbf{F}}_r}$$
$$j'_{Np,N} : X_0(Np)_{\bar{\mathbf{F}}_r} \to X_0(N)_{\bar{\mathbf{F}}_r}$$

are étale away from the cusps, so the divisors of $j^*\Delta$ and $j'^*\Delta$ on $X_0(Np)_{\bar{\mathbf{F}}_r}$ are the usual cuspidal divisors (2.3) and (2.4). As before it follows that the divisor of $f_p$ on $\mathfrak{T}_{p,\bar{\mathbf{F}}_r}$ is trivial.

We now consider the case $r = p$. Here we compute directly on $\mathfrak{T}_{p,\bar{\mathbf{F}}_p}$. By the Eichler-Shimura relation (1.3), we can write $\mathfrak{T}_{p,\bar{\mathbf{F}}_p} = \Gamma_p + \Gamma_p^{\#}$. Since we know that the divisor of $f_p$ is of codimension 0 in $\mathfrak{T}_{p,\bar{\mathbf{F}}_p}$, it suffices to compute it generically on the irreducible components $\Gamma_p$ and $\Gamma_p^{\#}$. Recall that $\Gamma_p$ is the base change to $\bar{\mathbf{F}}_p$ of the scheme-theoretic image of

$$\mathrm{id} \times \mathrm{Fr} : X_0(N)_{\mathbf{F}_p} \to X_0(N)_{\mathbf{F}_p} \times X_0(N)_{\mathbf{F}_p}.$$

Thus $\pi_1|_{\Gamma_p}$ is an isomorphism, so the divisor of $\pi_1^*\Delta$ has no support on $\Gamma_p$. However, $\pi_2|_{\Gamma_p}$ is purely inseparable, so by [**Sil86**, Chapter 2, Proposition 4.2(c)] the divisor of $\pi_2^*\Delta$ picks up 6 factors of $\Gamma_p$. (The 6 comes from the fact that $\Delta$ is an element of $\Omega^{\otimes 6}_{X_0(N)}$.) In the same way, the divisor of $\pi_1^*\Delta|_{\Gamma_p^\#}$ is $6\Gamma_p^\#$, while the divisor of $\pi_2^*\Delta|_{\Gamma_p^\#}$ is trivial. Combining these results, we find that $f_p$ has divisor $6(\Gamma_p^\# - \Gamma_p)$ in characteristic $p$. □

## 4. The cohesive Flach system

Given our analysis to this point, the proof of the following theorem is not difficult. Recall that we have assumed that $N$ is squarefree and that $l \geq 7$ is a prime not dividing $N$. $B$ is the image of $\mathbf{T}_0(N)$ in $\mathrm{End}_{\mathbf{Z}_l} H^1(X_0(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$ where we have assumed that all $T_p$ are self-adjoint. We have $A = B_\mathfrak{m}$ and

$$H = H^1\big(X_0(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l\big)_\mathfrak{m}$$

for $\mathfrak{m}$ a non-Eisenstein maximal ideal of $B$ associated to a newform. and we have fixed a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ with congruence element $\eta$.

THEOREM 4.1. *Let $H$ be a modular Galois representation as above and set $T = \mathrm{End}^0_A H(1)$. Assume that $T \otimes_A k$ is absolutely irreducible and that*

$$H^1(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}]) = 0$$

*for all ideals $\mathfrak{a}$ of finite index in $A$. Then $T$ admits a cohesive Flach system of Eichler-Shimura type of depth $\eta$ and weight $-12$ for the structure $\mathcal{S}_c$.*

PROOF. This is an immediate consequence of Theorem IX.6.1 and Lemma 3.1 once we check that the hypotheses of Theorem IX.6.1 are satisfied. $X_0(N)$ is cohomologically torsion-free at $l$ since it is a curve. The fact that $T_p^*$ is the trace of $\mathrm{Fr}_p$ on $H$ is [**DDT97**, Theorem 3.1] (taking into account that the representation there is the dual of ours and Frobenius there is arithmetic). This leaves the four numbered conditions of Section IX.4. By [**DDT97**, Lemma 3.27], $H$ is minimally ramified and ordinary; Lemma I.5.2 now gives condition 1 since $l \geq 7$. That $H \otimes_A k$ is absolutely irreducible follows from the fact that $\mathfrak{m}$ is non-Eisenstein, as discussed in Section 1.8; the rest of condition 2 is one of our hypotheses. Condition 3 is as well, and condition 4 is shown in [**DDT97**, Lemma 4.1]. Finally, the crystalline conditions come from Proposition VII.10.1 on checking that the cycle class of $T_p$ vanishes in $\mathrm{End}^0_A H$; that is, that it is a scalar. This is clear from the computations of Lemma IX.4.1 and the compatibility of the cycle class map with specialization as in [**GBI71**, Appendix to Exposé 10]. (The singularities of the Hecke correspondences are all ordinary double points, which can indeed be resolved over $\mathbf{Z}_l$.) This completes the proof. □

Note that the cohesive Flach system of Theorem 4.1 is canonically determined up to the choice of Gorenstein trace tr. Note also that the depth $\eta$ is canonically determined as the congruence element of tr.

CHAPTER 11

# The modular curve $X_1(N)$

In this chapter we construct cohesive Flach systems of Eichler-Shimura type for Galois representations associated to modular forms of weight 2 and arbitrary character. We assume for this chapter that $N$ is a squarefree integer greater than or equal to 7.

## 1. The geometry of $X_1(N)$

**1.1. The model $\mathfrak{X}_1(N)$.** Let $E/S$ be a generalized elliptic curve over a $\mathbf{Z}[\frac{1}{N}]$-scheme $S$. We define a $\Gamma_1(N)$-*structure* on $E/S$ to be a section $P : \mathbf{Z}/N\mathbf{Z} \hookrightarrow E$ of exact order $N$ on fibers; we further require that the subgroup generated by $P$ meet every irreducible component of fibers which are Néron polygons. (As before this implies that only Néron $d$-gons for $d$ dividing $N$ can support $\Gamma_1(N)$-structures.) We consider two $\Gamma_1(N)$-structures $(E/S, P)$ and $(E'/S, P')$ to be isomorphic if there is an isomorphism $E \xrightarrow{\simeq} E'$ taking $P$ to $P'$.

The functor from the category of $\mathbf{Z}[\frac{1}{N}]$-schemes to sets sending a scheme $S$ to the set of isomorphism classes of $\Gamma_1(N)$-structures on generalized elliptic curves over $S$ is representable (only coarsely representable for $N \leq 4$) by a scheme $\mathfrak{X}_1(N)$. $\mathfrak{X}_1(N)$ is a proper, smooth, geometrically connected $\mathbf{Z}[\frac{1}{N}]$-scheme of relative dimension 1. $\mathfrak{X}_1(N)$ admits a proper, regular model over $\mathbf{Z}$ as well; see [**DI95**, Sections 8.2, 8.3, 9.2, 9.3].

**1.2. The degeneracy maps.** For all $N$ dividing $M$, there is a natural *degeneracy map*

$$j_{M,N} : \mathfrak{X}_1(M) \to \mathfrak{X}_1(N);$$

here we are using a model for $\mathfrak{X}_1(M)$ defined over $\mathbf{Z}[\frac{1}{N}]$. This map is defined on the moduli level by sending the $\Gamma_1(M)$-structure $(E/S, P)$ to the $\Gamma_1(N)$-structure $(E/S, \frac{M}{N}P)$.

**1.3. The diamond automorphisms.** For $d \in (\mathbf{Z}/N\mathbf{Z})^\times$ there is an automorphism of the above moduli problem sending a pair $(E/S, P)$ to the pair $(E/S, dP)$; the corresponding automorphism of $\mathfrak{X}_1(N)$ is called a *diamond automorphism* and is written $\langle d \rangle$. Note that $\langle -1 \rangle$ acts trivially on $\mathfrak{X}_1(N)$, and that $(\mathbf{Z}/N\mathbf{Z})^\times/\pm 1$ acts freely on $\mathfrak{X}_1(N)$.

$\mathfrak{X}_0(N)$ is defined to be the quotient of $\mathfrak{X}_1(N)$ by $(\mathbf{Z}/N\mathbf{Z})^\times/\pm 1$; the natural quotient map

$$\pi_N : \mathfrak{X}_1(N) \to \mathfrak{X}_0(N)$$

realizes $\mathfrak{X}_1(N)$ as a finite Galois covering of $\mathfrak{X}_0(N)$, of degree $\phi(N)/2$.

**1.4. The cusps.** The closed subschemes of $\mathfrak{X}_1(N)$ corresponding to families containing Néron polygons are called the *cusps*. The map $\pi_N$ is a finite Galois covering unramified at the cusps, so over any algebraically closed field $k$ there are exactly $\phi(N)/2$ cusps of $\mathfrak{X}_1(N)$ sitting over each cusp of $\mathfrak{X}_0(N)$.

**1.5. The Hecke correspondences.** Fix a prime $p$ not dividing $N$. Let $E/S$ be a generalized elliptic curve over a $\mathbf{Z}[\frac{1}{N}]$-scheme $S$. We define a $\Gamma_1(N;p)$*-structure* on $E/S$ to be a pair of a $\Gamma_1(N)$-structure $P$ and a finite flat subgroup scheme $C$ of $E$ with all geometric fibers of rank $p$; we require that the group generated by $P$ and $C$ meet every irreducible component of Néron polygon fibers. The $\Gamma_1(N;p)$-moduli problem is representable by a proper, regular, geometrically irreducible $\mathbf{Z}[\frac{1}{N}]$-scheme $\mathfrak{X}_1(N;p)$ of relative dimension 1; it becomes smooth over $\mathbf{Z}[\frac{1}{Np}]$. (See [**DI95**, Sections 8.3 and 9.3].)

$\mathfrak{X}_1(N;p)$ admits two natural degeneracy maps

$$j_{N;p,N} : \mathfrak{X}_1(N;p) \to \mathfrak{X}_1(N)$$
$$j'_{N;p,N} : \mathfrak{X}_1(N;p) \to \mathfrak{X}_1(N).$$

$j_{N;p,N}$ sends the triple $(E/S, P, C)$ to the pair $(E/S, P)$, and $j'_{N;p,N}$ sends it to $((E/C)/S, P)$. These maps are both generically étale away from characteristic $p$.

We define the $p^{th}$ *Hecke correspondence* $\mathfrak{T}_p$ to be the scheme-theoretic image of the map

$$j_{N;p,N} \times j'_{N;p,N} : \mathfrak{X}_1(N;p) \to \mathfrak{X}_1(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_1(N).$$

$\mathfrak{T}_p$ is birational to $\mathfrak{X}_1(N;p)$ away from characteristic $p$, and has pure codimension 1 in $\mathfrak{X}_1(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_1(N)$.

We can give a precise description of the closed subscheme $\mathfrak{T}_{p,\mathbf{F}_p}$ of the proper smooth variety $\mathfrak{X}_1(N)_{\mathbf{F}_p} \times_{\operatorname{Spec} \mathbf{F}_p} \mathfrak{X}_1(N)_{\mathbf{F}_p}$. We will see later that $\mathfrak{T}_{p,\mathbf{F}_p}$ can be considered as an algebraic self-correspondence on $\mathfrak{X}_1(N)_{\mathbf{F}_p}$; the *Eichler-Shimura relation* states that

$$\mathfrak{T}_{p,\mathbf{F}_p} = \Gamma_p + \Gamma'_p$$

where $\Gamma_p$ is the graph of the Frobenius morphism on $\mathfrak{X}_1(N)_{\mathbf{F}_p}$ and $\Gamma'_p$ is its modified transpose given as the image of

$$\mathrm{Fr} \times \langle p \rangle : \mathfrak{X}_1(N)_{\mathbf{F}_p} \to \mathfrak{X}_1(N)_{\mathbf{F}_p} \times \mathfrak{X}_1(N)_{\mathbf{F}_p}.$$

See [**Gro90**, p. 454] and [**DI95**, Section 8.4].

**1.6. The Atkin correspondences.** Fix a prime $p$ dividing $N$. Let $E/S$ be a generalized elliptic curve over a $\mathbf{Z}[\frac{1}{N}]$-scheme $S$. We define a $\Gamma_1(N;p)$*-structure* on $E/S$ to be a pair of a $\Gamma_1(N)$-structure $P$ and a finite flat subgroup scheme $C$ of $E$ with all geometric fibers of rank $p$; we require that the group generated by $P$ and $C$ meet every irreducible component of Néron polygon fibers and we further require that $C$ has trivial intersection with the group generated by $P$. The $\Gamma_1(N;p)$-moduli problem is representable by a proper, regular $\mathbf{Z}[\frac{1}{N}]$-scheme $\mathfrak{X}_1(N;p)$ with geometrically irreducible fibers and of relative dimension 1; it becomes smooth over $\mathbf{Z}[\frac{1}{Np}]$. See [**Gro90**, p. 454].

$\mathfrak{X}_1(N;p)$ admits two natural degeneracy maps $j_{N;p,N}$ and $j'_{N;p,N}$ to $\mathfrak{X}_1(N)$: the first sends the triple $(E/S, P, C)$ to the pair $(E/S, P)$ and the second sends $(E/S, P, C)$ to $((E/C)/S, P)$; here $P$ denotes the induced section of $E/C$, which still had order $N$ since $C$ is not contained in the group generated by $P$. We define the $p^{th}$ *Atkin correspondence* $\mathfrak{T}_p$ to be the scheme-theoretic image of the map

$$j_{N;p,N} \times j'_{N;p,N} : \mathfrak{X}_1(N;p) \to \mathfrak{X}_1(N) \times_{\mathrm{Spec}\,\mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_1(N).$$

As before, $\mathfrak{T}_p$ is birational to $\mathfrak{X}_1(N;p)$ away from characteristic $p$ and has pure codimension 1 in $\mathfrak{X}_1(N) \times_{\mathrm{Spec}\,\mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_1(N)$. See [**MW84**, Section 5.5] for more details.

There is a natural projection $\pi'_N : \mathfrak{X}_1(N;p) \to \mathfrak{X}_0(N;p)$; in fact, it is simply the quotient map by the diamond automorphisms (acting on the point of order $N$) and is a Galois covering of degree $\phi(N)/2$. It is unramified at the cusps, so it allows us to understand the cusps on $\mathfrak{X}_1(N;p)$ in terms of the cusps of $\mathfrak{X}_0(N;p)$.

**1.7. The Hecke algebra $\mathbf{T}_1(N)$.** The modular curve $X_1(N)$ is the generic fiber $\mathfrak{X}_1(N) \times_{\mathrm{Spec}\,\mathbf{Z}[\frac{1}{N}]} \mathrm{Spec}\,\mathbf{Q}$ of $\mathfrak{X}_1(N)$; it is a smooth projective curve over $\mathbf{Q}$. We regard the diamond automorphisms $\langle d \rangle$ as algebraic correspondences on $\mathfrak{X}_1(N)$ via their graphs. Write $T_p$ for the Hecke correspondence $\mathfrak{T}_{p,\mathbf{Q}}$ on $X_1(N)$. One can also define Hecke correspondences $T_n$ for all $n$ as with $X_0(N)$. The same argument as in the $X_0(N)$ case shows that these are all algebraic correspondences in our sense. Note also that the compositions $\langle d \rangle \circ T_n$ and $T_n \circ \langle d \rangle$ are trivially defined since $\langle d \rangle$ is an automorphism.

The Hecke correspondences satisfy the relations

$$\begin{aligned} T_{mn} &= T_m T_n && \text{for } m,n \text{ relatively prime;}\\ T_{p^n} &= T_{p^{n-1}} T_p - p\,\langle p \rangle\, T_{p^{n-2}} && \text{for } p \text{ not dividing } N;\\ T_{p^n} &= T_p^n && \text{for } p \text{ dividing } N; \end{aligned}$$

see [**Lan76**, Chapter 7, Theorem 2.1]. The $T_p$ and diamond automorphisms generate a commutative algebra of correspondences which we denote $\mathbf{T}_1(N)$. We can omit any $T_l$ from the set of generators of $\mathbf{T}_1(N)$ so long as $l$ does not divide $N$ by [**DDT97**, Lemma 4.1].

**1.8. The involution $w_\zeta$.** We will need one "exotic" involution of $X_1(N)_{\bar{\mathbf{Q}}}$. Fix a primitive $N^{\mathrm{th}}$ root of unity $\zeta$ and for every elliptic curve $E/S$ over a $\mathbf{Z}[\frac{1}{N}]$-scheme $S$, let $e : E[N] \times E[N] \to \mu_N$ be the scheme-theoretic Weil pairing. $w_\zeta$ is the automorphism of $\mathfrak{X}_1(N)_{\mathbf{Z}[1/N,\zeta]}$ sending a pair $(E/S, P)$ to the pair $((E/C)/S, Q)$, where $C$ is the subgroup of $E$ generated by $P$ and $Q$ is the unique point of $E[N]$ such that $e(Q,P) = \zeta$; see [**MW84**, Section 5.2] for more details.

Consider now the corresponding involution of $X_1(N)_{\bar{\mathbf{Q}}}$, which we also write as $w_\zeta$. $w_\zeta$ is self-adjoint in the sense that $w_\zeta^* = w_{\zeta *}$ acting on cohomology. $w_\zeta$ interacts with elements of $\mathbf{T}_1(N)$ via the relation $\alpha^* w_\zeta^* = w_\zeta^* \alpha_*$. Since $w_\zeta$ is an involution, this implies also that $\alpha_* w_\zeta^* = w_\zeta^* \alpha^*$; see [**Til97**].

We also have a simple interaction with the Galois action on étale cohomology: for $v \in H^1(X_1(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$ and $\sigma \in G_{\mathbf{Q}}$, we have (considering $w_\zeta^*$ as an involution on cohomology)

$$\sigma(w_\zeta^* v) = (\sigma w_\zeta^*)(\sigma v) = \langle \sigma \rangle^{*\,-1} w_\zeta^* (\sigma v).$$

Here we are using the natural identification of $\mathrm{Gal}(\mathbf{Q}(\zeta_N)^+/\mathbf{Q})$ with $(\mathbf{Z}/N\mathbf{Z})^\times/\pm 1$ to regard $\langle \cdot \rangle$ as a character of $G_{\mathbf{Q}}$. See [**MT73**, Section 2] for details.

**1.9. Poincaré duality.** Fix a prime $l \geq 7$ and not dividing $N$ and let $V$ be the étale cohomology group $H^1(X_1(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$. The algebra $\mathbf{T}_1(N)$ is not self-adjoint with respect to the Poincaré pairing $\varphi : V \otimes_{\mathbf{Z}_l} V(1) \to \mathbf{Z}_l$. Since $w_\zeta$ is self-adjoint, $\varphi$ does satisfy $\varphi(w_\zeta^* t, t') = \varphi(t, w_\zeta^* t')$.

Let $B_*$ and $B^*$ be the images of $\mathbf{T}_1(N)$ in $\operatorname{End}_{\mathbf{Z}_l} V$, as usual. We define an untwisting of $\mathbf{T}_1(N)$ as follows: let $\tilde{B}$ be a free $B^*$-module of rank 1 (with a chosen generator $\xi$) with a $B^*$-linear action of $G_{\mathbf{Q}}$ given by

$$\sigma\xi = \langle\sigma\rangle^* \xi.$$

We claim that the map

$$\xi \otimes w_\zeta^* : V \to \tilde{B} \otimes_{B^*} V$$

sending $v$ to $\xi \otimes w_\zeta^* v$ is Galois equivariant. Indeed,

$$\begin{aligned}\sigma\big(\xi \otimes w_\zeta^*(v)\big) &= \sigma\xi \otimes \sigma\big(w_\zeta^*(v)\big) \\ &= \langle\sigma\rangle^* \xi \otimes \langle\sigma\rangle^{*\,-1} w_\zeta^*(\sigma v) \\ &= \xi \otimes w_\zeta^*(\sigma v) \\ &= (\xi \otimes w_\zeta^*)(\sigma v).\end{aligned}$$

Thus the triple $(w_\zeta^*, \tilde{B}, \xi)$ is an untwisting of $V$.

**1.10. Non-Eisenstein maximal ideals.** The theory of maximal ideals in $\mathbf{T}_1(N)$ is very similar to that of $\mathbf{T}_0(N)$ in Section X.1.8; see [**DDT97**, Chapter 4] and [**Til97**, Theorem 3.4]. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $B^*$ associated to a newform at level $N$. Set $A = B^*_{\mathfrak{m}}$, $k = A/\mathfrak{m}_A$, $H = V \otimes_{B^*} A$ and $\tilde{A} = \tilde{B} \otimes_{B^*} A$. $H \otimes_A k$ is absolutely irreducible as a $G_{\mathbf{Q}}$-module, and as usual we know that $A$ is Gorenstein. Fix a choice of Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$ with congruence element $\eta$. We set $T = \operatorname{End}^0_A H$ and we endow $T$ with the finite/singular structure which is unramified away from $Nl$, minimally ramified at $N$ and crystalline at $l$.

## 2. Admissible markings

We once again will use $\Delta$ as our admissible marking.

LEMMA 2.1. *The marking $\Delta$ is admissible for the Hecke algebra $\mathbf{T}_1(N)$.*

PROOF. Since the maps $X_1(N;p) \to X_0(N;p)$ are finite Galois coverings unramified at the cusps, the verification that $\Delta$ is divisorless for all $T_p$ follows immediately from Lemma X.2.1. It remains to check that $\Delta$ is divisorless for the diamond automorphisms $\langle d\rangle$. The induced rational function on the correspondence $\langle d\rangle$ (which is isomorphic to $X_1(N)$) is simply $\Delta \circ \langle d\rangle / \Delta$; since $\Delta$ has trivial character, this is the constant 1. Thus $\langle d\rangle$ is divisorless for $\Delta$, and we conclude that all of $\mathbf{T}_1(N)$ is divisorless for $\Delta$. □

Let $f_p$ denote the rational function on $T_p$ induced by $\Delta$. The calculation of the divisor of $f_p$ in positive characteristic is entirely similar to that of Lemma X.3.1, taking into account the modification in the Eichler-Shimura relation and the fact that $\langle p\rangle$ is étale. One finds the following result; here $\Gamma_p$ is the graph of Frobenius and $\Gamma'_p$ is its modified transpose as in Section 1.5.

LEMMA 2.2. *For all $p$ not dividing $N$, the pair $(T_p, f_p)$ is a divisorial lifting of $6(\Gamma'_p - \Gamma_p)$.*

## 3. The cohesive Flach system

Recall that we have assumed that $N \geq 5$ is squarefree and that $l \geq 7$ is a prime not dividing $N$. $B^*$ is the image of $\mathbf{T}_1(N)$ in $\mathrm{End}_{\mathbf{Z}_l} H^1(X_1(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$. We have $A = B^*_{\mathfrak{m}}$ and

$$H = H^1\big(X_0(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l\big)_{\mathfrak{m}}$$

for $\mathfrak{m}$ a non-Eisenstein maximal ideal of $B^*$ associated to a newform, and we have fixed a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ with congruence element $\eta$.

THEOREM 3.1. *Let $H$ be a modular Galois representation as above and set $T = \mathrm{End}^0_A H(1)$. Assume that $T \otimes_A k$ is absolutely irreducible and that*

$$H^1\big(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}]\big) = 0$$

*for all ideals $\mathfrak{a}$ of finite index in $A$. Then $T$ admits a cohesive Flach system of Eichler-Shimura type of depth $\eta$ and weight $-12$.*

PROOF. The fact that the diamond automorphisms are diamond operators in the sense of Definition IX.5.1 follows from [**DDT97**, Theorem 3.1]. The rest of the proof is virtually identical to the proof of Theorem X.4.1. The one complication is the check that the minimally ramified structure agrees with the weak structure. For this one uses [**Car86**, Théorème A] to see that $H$ is either ordinary or else a direct sum of an unramified character and a tamely ramified character. The first case is dealt with via Lemma I.5.2, while the second case is a straightforward computation. □

CHAPTER 12

# Kuga-Sato varieties

In this chapter we extend the methods of the previous two chapters to construct cohesive Flach systems of Eichler-Shimura type for modular Galois representations of higher weight. There are many possible approaches to this. We choose the least elegant but simplest geometrically: we realize these representations in the cohomology of an "open" Kuga-Sato variety. This requires some modifications of the results of Chapters VII and VIII, but has the advantage of not involving any resolution of singularities.

## 1. The geometry of Kuga-Sato varieties

**1.1. The universal elliptic curve.** Recall that $\mathfrak{X}_1(N)$ represents the $\Gamma_1(N)$-moduli problem for $\mathbf{Z}[\frac{1}{N}]$-schemes. We will be interested only in the complement $\mathcal{Y}_1(N)$ of the cusps in $\mathfrak{X}_1(N)$; $\mathcal{Y}_1(N)$ is a smooth separated $\mathbf{Z}[\frac{1}{N}]$-scheme, but it is not proper. $\mathcal{Y}_1(N)$ represents the $\Gamma_1(N)$-moduli problem for elliptic curves over $\mathbf{Z}[\frac{1}{N}]$; in particular, there exists a universal elliptic curve (with a point of exact order $N$) $\mathcal{E}_1(N)$ over $\mathcal{Y}_1(N)$.

**1.2. Open Kuga-Sato schemes.** For $k \geq 0$, we let $\mathcal{E}_1^k(N)$ denote the $k$-fold fiber product of $\mathcal{E}_1(N)$ over $\mathcal{Y}_1(N)$:

$$\mathcal{E}_1^k(N) = \overbrace{\mathcal{E}_1(N) \times_{\mathcal{Y}_1(N)} \mathcal{E}_1(N) \times \cdots \times_{\mathcal{Y}_1(N)} \mathcal{E}_1(N)}^{k}.$$

$\mathcal{E}_1^k(N) \to \mathbf{Z}[\frac{1}{N}]$ is smooth and separated of relative dimension $k+1$, but is not proper. We call $\mathcal{E}_1^k(N)$ the *open Kuga-Sato scheme of weight* $k+2$. $\mathcal{E}_1^k(N)$ has a canonical compactification in each characteristic (see [**Del71**, Lemma 5.4] and [**Con**, Theorem 4.3.1.1]) but (with one exception) we will not need this.

**1.3. The Hecke algebra.** For $d \in (\mathbf{Z}/N\mathbf{Z})^\times$ there is an automorphism $\langle d \rangle$ of $\mathcal{E}_1(N)$ sitting over the automorphism $\langle d \rangle$ of $\mathcal{Y}_1(N)$; see [**Con**, Section 4.2.7] where it is denoted $I_d$. This in turn yields an automorphism of $\mathcal{E}_1^k(N)$ which we denote by $\langle d \rangle^{(k)}$. Similarly, the Hecke correspondences $\mathfrak{T}_p$ (for all $p$ not dividing $N$) on $\mathcal{Y}_1(N)$ yield (by base change) Hecke correspondences $\mathfrak{T}_p^{(k)}$ on $\mathcal{E}_1^k(N)$. (One can also obtain these operators by a generalization of the methods of Section XI.1.5; see [**Sch90**, Section 4].)

We define $E_1^k(N)$ to be the generic fiber of $\mathcal{E}_1^k(N)$; it is a smooth separated variety of dimension $k+1$ over $\mathbf{Q}$. Let $T_p^{(k)}$ denote the Hecke correspondence $\mathfrak{T}_{p,\mathbf{Q}}^{(k)}$ on $E_1^k(N)$. We note that the projections $T_{p,\mathbf{Q}}^{(k)} \to E_1^k(N)$ are finite flat of the same degree (by base change), although $T_{p,\mathbf{Q}}^{(k)}$ is not technically an algebraic correspondence in the sense of Chapter VIII as $E_1^k(N)$ is not proper over $\mathbf{Q}$. However, $T_p^{(k)}$

still yields maps in $K$-theory and the notion of composition as in Section VIII.3 is still valid in this setting. In particular, it follows from the relations of [**Lan76**, Chapter 7, Theorem 2.1] that the $T_p^{(k)}$ and $\langle d\rangle^{(k)}$ generate a commutative algebra of correspondences for the open variety $E_1^k(N)$; we denote this algebra by $\mathbf{T}_1(N)^{(k)}$. (Note that we have not yet defined any maps on étale cohomology associated to these correspondences.) As always we can omit any $T_l^{(k)}$ from the generators of $\mathbf{T}_1(N)^{(k)}$ so long as $l$ does not divide $N$.

Let $p$ be a prime not dividing $N$. For such a $p$ we have the usual Eichler-Shimura relation for $\mathfrak{T}_{p,\mathbf{F}_p}$, regarded as an algebraic correspondence on the smooth variety $\mathcal{E}_1^k(N)_{\mathbf{F}_p} \times_{\operatorname{Spec}\mathbf{F}_p} \mathcal{E}_1^k(N)_{\mathbf{F}_p}$:

$$\mathfrak{T}_{p,\mathbf{F}_p} = \Gamma_p + \Gamma_p' \tag{1.1}$$

where $\Gamma_p$ is the graph of the Frobenius morphism on $\mathcal{E}_1^k(N)_{\mathbf{F}_p}$ and $\Gamma_p'$ is its modified transpose given as the image of

$$\mathrm{Fr}\times\langle p\rangle : \mathcal{E}_1^k(N)_{\mathbf{F}_p} \to \mathcal{E}_1^k(N)_{\mathbf{F}_p} \times_{\operatorname{Spec}\mathbf{F}_p} \mathcal{E}_1^k(N)_{\mathbf{F}_p}.$$

See [**Con**, Theorem 5.3.3.1] for details.

**1.4. Untwistings.** Let $\zeta$ denote a fixed primitive $N^{\text{th}}$ root of unity. By [**Con**, Section 4.2.7], the involution $w_\zeta$ of $\mathcal{Y}_1(N)_{\mathbf{Z}[1/N,\zeta]}$ lifts to an involution $w_\zeta$ (called $\varphi_\zeta$ in [**Con**]) of $\mathcal{E}_1(N)_{\mathbf{Z}[1/N,\zeta]}$; this in turn yields an involution $w_\zeta^{(k)}$ of $\mathcal{E}_1^k(N)_{\mathbf{Z}[1/N,\zeta]}$.

**1.5. Galois representations.** The production of Galois representations from $V$ and maximal ideals of $B^*$ is due to Deligne; see [**Del71**]. We follow the presentation of [**Con**] via the integral theory of [**FJ95**].

Fix a prime $l \geq \max\{7, k+1\}$ not dividing $N$. We consider first the image

$$V = \tilde{H}^1\big(Y_1(N)_{\bar{\mathbf{Q}}}, \operatorname{Sym}^k R^1 f_*\mathbf{Z}_l\big)$$

of the map

$$H_c^1\big(Y_1(N)_{\bar{\mathbf{Q}}}, \operatorname{Sym}^k R^1 f_*\mathbf{Z}_l\big) \to H^1\big(Y_1(N)_{\bar{\mathbf{Q}}}, \operatorname{Sym}^k R^1 f_*\mathbf{Z}_l\big)$$

of compactly supported cohomology into ordinary cohomology. Here $f : E_1(N)_{\bar{\mathbf{Q}}} \to Y_1(N)_{\bar{\mathbf{Q}}}$ is the structure map. By [**Con**, Section 5.4], [**Sch90**, Proposition 4.1.1] and [**Car94**] (which discusses the integral structure), $V$ is a subquotient of the étale cohomology group $\tilde{H}^{k+1}(E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$. We must assume that:

- $E_1^k(N)$ is cohomologically torsion-free at $l$;
- $V$ is a direct summand of $\tilde{H}^{k+1}(E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$ as a $\mathbf{Z}_l$-module.

Both conditions hold for almost all $l$.

Given these assumptions, we claim that we can apply the methods of Chapters VII and VIII for the open variety $E_1^k(N) \times_{\operatorname{Spec}\mathbf{Q}} E_1^k(N)$ to the direct summand (via the above assumptions and the Künneth formula) $\operatorname{End}_{\mathbf{Z}_l} V$ inside of

$$V_0 = H^{2k+2}\big(E_1^k(N)_{\bar{\mathbf{Q}}} \times_{\operatorname{Spec}\bar{\mathbf{Q}}} E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l\big). \tag{1.2}$$

To show this we must consider where proper hypotheses were used in these chapters. Properness was used in three places in Chapter VII. In Section VII.3 it was used to guarantee the local constancy of a higher direct image for an application of purity; however, [**Ras89**, Lemma 2.1] applies even without this, so that here properness was not required. The critical application of properness was in Section VII.7 where it was used to relate the cohomology of the generic fiber with the cohomology of

the special fiber. By [**Con**, Theorem 5.2.8.1], our $V$ is compatible with such base change, and one sees immediately that this compatibility is compatible with the inclusion of $V$ into $H^{k+1}(E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$. This is sufficient to extend Theorem VII.1.1 to apply to $\mathrm{End}_{\mathbf{Z}_l} V$ inside of (1.2); that is, there is a commutative diagram
(1.3)

$$\begin{array}{ccc}
E_2^{k+1,-k-2}\big(E_1^k(N) \times E_1^k(N)\big) & \xrightarrow{\mathrm{div}_{Fp}} & A^{k+1}\big(\mathcal{E}_1^k(N)_{\mathbf{F}_p} \times \mathcal{E}_1^k(N)_{\mathbf{F}_p}\big) \\
\downarrow{\scriptstyle \sigma_{k+1}} & & \downarrow{\scriptstyle s} \\
H^1(\mathbf{Q}, V_0) & & H^{2k+2}\big(\mathcal{E}_1^k(N)_{\bar{\mathbf{F}}_p} \times \mathcal{E}_1^k(N)_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(k+1)\big)^{G_{\mathbf{F}_p}} \\
\downarrow & & \downarrow{\scriptstyle \simeq} \\
H^1(\mathbf{Q}, \mathrm{End}_{\mathbf{Z}_l} V) & & (\mathrm{End}_{\mathbf{Z}_l})^{G_{\mathbf{F}_p}} \\
\downarrow & & \downarrow \\
H^1(\mathbf{Q}_p, \mathrm{End}_{\mathbf{Z}_l} V) & \longrightarrow & H^1_s(\mathbf{Q}_p, \mathrm{End}_{\mathbf{Z}_l} V)
\end{array}$$

for each $p$ not dividing $Nl$.

The last application of properness in Chapter VII occurred in Section 10 where it was assumed of the ambient variety. However, the deRham cases of these results remain valid without this properness, and in our particular case the crystalline case remains valid by the explicit compactification of $E_1^k(N)$ given by Kuga-Sato varieties.

In Chapter VIII properness was used frequently, but always to insure that appropriate operations in étale cohomology (covariant functoriality, Poincaré duality and Künneth projections) operate entirely on ordinary cohomology (that is, without introducing compact cohomology). The results of [**Con**, Section 5.2.1.1], [**Sch90**, Proposition 4.1.1] and especially [**FJ95**, Theorem 2.1] show that these operations for the Hecke algebra $\mathbf{T}_1(N)^{(k)}$ can be regarded as operating entirely on $V$. We conclude that the results of Chapter VIII, and thus the results of Chapter IX, remain valid for the Flach map

$$E_2^{k+1,-k-2}\big(E_1^k(N) \times_{\mathrm{Spec}\,\mathbf{Q}} E_1^k(N)\big) \to H^1(\mathbf{Q}, \mathrm{End}_{\mathbf{Z}_l} V)$$

of (1.3).

We let $B_*$ and $B^*$ denote the images of $\mathbf{T}_1(N)^{(k)}$ in $\mathrm{End}_{\mathbf{Z}_l} V$. We define an untwisting of $V$ via $w_\zeta^{(k)}$ exactly as in Section XI.1.9, using [**FJ95**] to check that it is an untwisting. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $B^*$ associated to a newform; by [**FJ95**, Theorem 2.1 and Theorem 3.38], such an $\mathfrak{m}$ is dualizing. In particular, $H = V_{\mathfrak{m}}$ is free of rank 2 over the Gorenstein ring $A = B^*_{\mathfrak{m}}$ and $H \otimes_A k$ is absolutely irreducible. We fix a choice of Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ and let $\eta \in A$ be the associated congruence element. We set $T = \mathrm{End}_A^0 H(1)$ and give it the finite/singular structure which is unramified away from $Nl$, minimally ramified at $N$ and crystalline at $l$.

## 2. Admissible markings

Note that by pullback the modular form $\Delta$ yields a differential form on $E_1^k(N)$. We use $\Delta$ as a marking for the Hecke algebra $\mathbf{T}_1(N)^{(k)}$. It is actually possible in this setting to use any modular unit as a marking, but we will not pursue this here.

LEMMA 2.1. *The marking* $\Delta$ *is admissible for the Hecke algebra* $\mathbf{T}_1(N)^{(k)}$.

PROOF. Recall that $T_p^{(k)}$ is obtained by pullback from the Hecke correspondence $T_p$ on $Y_1(N)$. By the definition of the rational function induced by a marking, we see that the rational function $f_p^{(k)}$ on $T_p^{(k)}$ induced by $\Delta$ is precisely the pullback of the rational function $f_p$ on $T_p$ induced by $\Delta$. Since by Lemma XI.2.1 we know that $f_p$ has trivial divisor on $T_p$, we see that $T_p^{(k)}$ is divisorless for $\Delta$. The proof for the diamond automorphisms is entirely similar. □

LEMMA 2.2. *For all* $p$ *not dividing* $N$*, the pair* $(T_p^{(k)}, f_p^{(k)})$ *on* $E_1^k(N) \times E_1^k(N)$ *is a divisorial lifting of the cycle* $6(\Gamma_p' - \Gamma_p)$ *on* $\mathcal{E}_1^k(N)_{\mathbf{F}_p} \times_{\operatorname{Spec} \mathbf{F}_p} \mathcal{E}_1^k(N)_{\mathbf{F}_p}$.

PROOF. The proof of this is identical to the proof of Lemma XI.3.1 using the Eichler-Shimura relation (1.1) in this context and regarding $\Delta$ as a differential form on $\mathcal{E}_1^k(N)$; see also [**Con**, Theorem 5.3.3.1]. □

## 3. The cohesive Flach system

Recall that we have assumed that $N \geq 5$ is squarefree and that $l \geq \max\{7, k+1\}$ is a prime not dividing $N$. We have also assumed that

- $E_1^k(N)$ is cohomologically torsion-free at $l$;
- $V$ is a direct summand of $\tilde{H}^{k+1}(E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$ as a $\mathbf{Z}_l$-module.

$B^*$ is the image of $\mathbf{T}_1(N)^{(k)}$ in $\operatorname{End}_{\mathbf{Z}_l} V$ for

$$V = \tilde{H}^1\big(Y_1(N)_{\bar{\mathbf{Q}}}, \operatorname{Sym}^k R^1 f_* \mathbf{Z}_l\big)$$

regarded as a direct summand of $H^{k+1}(E_1^k(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_l)$. We have $A = B^*_{\mathfrak{m}}$ and $H = V_{\mathfrak{m}}$ for $\mathfrak{m}$ a non-Eisenstein maximal ideal of $B^*$ associated to a newform, and we have fixed a Gorenstein trace $\operatorname{tr} : A \to \mathbf{Z}_l$ with congruence element $\eta$.

THEOREM 3.1. *Let* $H$ *be a modular Galois representation as above and set* $T = \operatorname{End}_A^0 H(1)$*. Assume that* $T \otimes_A k$ *is absolutely irreducible and that*

$$H^1\big(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}]\big) = 0$$

*for all ideals* $\mathfrak{a}$ *of finite index in* $A$*. Then* $T$ *admits a cohesive Flach system of Eichler-Shimura type of depth* $\eta$ *and weight* $-12$.

PROOF. The proof of this is virtually identical to the proof of Theorem XI.3.1. In particular, the crystalline condition again follows from Proposition VII.10.1 since the cycle class of $T_p$ is easily seen to be scalar. □

## 4. Applications

As observed in [**Fla92**], results like Theorem 3.1 can also be used to show that certain deformation problems are unobstructed.

THEOREM 4.1. *Let* $H$ *and* $T$ *be as above; in particular, we assume that* $l$ *does not divide* $N$*. Let* $S$ *denote the set of places of* $\mathbf{Q}$ *dividing* $Nl$ *together with the archimedean place. Assume that:*

- $T \otimes_A k$ *is absolutely irreducible;*
- $H^1\big(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}]\big) = 0$ *for all ideal* $\mathfrak{a}$ *of finite index in* $A$*;*

- *$\eta = A$;*
- *$H^0\big(\mathbf{Q}_p, \mathrm{End}_k^0(H \otimes_A k)(1)\big) = 0$ for all $p$ dividing $Nl$.*

*Then the deformation problem (with fixed determinant) for the residual representation*

$$\rho : G_{\mathbf{Q}_S} \to \mathrm{Aut}_k(H \otimes_A k)$$

*is cohomologically unobstructed. In particular, the associated universal deformation ring is a power series ring in two variables over $W(k)$.*

PROOF. This is derived from Theorem 3.1 as in [**Fla92**, Section 3]; see also [**Wes00**, Sections 3–5]. We are forced to impose the determinant condition as the Shafarevich-Tate group of the determinant of $H \otimes_A k$ is difficult to control if $H$ does not have cyclotomic determinant. $\square$

# Appendix

APPENDIX A

# Edge maps of spectral sequences

In this appendix we prove various compatibility results for edge maps of spectral sequences which are used in the text.

## 1. Notation for filtered complexes

We will use the notation of [**Wei94**, Chapter 5]; for clarity we review what we will need. We work only in the level of generality which we will need for the applications. Note that we can afford to be a bit carefree, as we already know that everything we are looking at is well-defined. We will also ignore all categorical issues without further comment.

Let $C^\bullet$ be a cochain complex supported in non-negative degree. We will write $d$ for the differential on $C^\bullet$. We assume that $C^\bullet$ has a filtration $F^\bullet C^\bullet$:

$$\cdots \subseteq F^{n+1}C^\bullet \subseteq F^nC^\bullet \subseteq \cdots \subseteq F^1C^\bullet \subseteq F^0C^\bullet = C^\bullet.$$

We assume that this filtration is canonically bounded: recall that this means that $F^{n+1}C^n = 0$ for each $n$, so that we have filtrations

$$0 = F^{n+1}C^n \subseteq F^nC^n \subseteq \cdots \subseteq F^1C^n \subseteq F^0C^n = C^n.$$

We associate a spectral sequence to this data as follows. For each $p, q, r \geq 0$, set

$$A_r^{pq} = \{c \in F^pC^{p+q} \mid d(c) \in F^{p+r}C^{p+q+1}\}.$$

Note that from this definition the differential yields a map

$$d : A_r^{pq} \to A_r^{p+r,q-r+1}. \tag{1.1}$$

Set $E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$ and let $\eta_{pq} : F^pC^{p+q} \to E_0^{pq}$ be the quotient map. Define

$$Z_r^{pq} = \eta_{pq}(A_r^{pq})$$
$$B_r^{pq} = \eta_{pq}\big(d(A_{r-1}^{p-r+1,q+r-2})\big)$$

for $r \geq 0$. Set $Z_\infty^{pq} = \cap_r Z_r^{pq}$ and $B_\infty^{pq} = \cup_r B_r^{pq}$. We have inclusions

$$0 = B_0^{pq} \subseteq B_1^{pq} \subseteq \cdots \subseteq B_\infty^{pq} \subseteq Z_\infty^{pq} \subseteq \cdots \subseteq Z_1^{pq} \subseteq Z_0^{pq} = E_0^{pq}.$$

Set

$$E_r^{pq} = Z_r^{pq}/B_r^{pq};$$

one checks immediately that this agrees with our previous definition for $r = 0$. The maps (1.1) above now yield maps

$$d : E_r^{pq} \to E_r^{p+r,q-r+1}$$

and one checks that this yields a spectral sequence

$$E_0^{pq} \Rightarrow H^{p+q}(C^\bullet). \tag{1.2}$$

Recall that convergence of (1.2) means that for each $p, q$ there is an isomorphism

$$E_\infty^{pq} \cong F^p H^{p+q}/F^{p+1}H^{p+q}. \tag{1.3}$$

One checks immediately that (1.3) is induced by the inclusion

$$A_\infty^{pq} \hookrightarrow F^p C^{p+q} \tag{1.4}$$

where $A_\infty^{pq}$ is defined to be $\cap_r A_r^{pq}$.

## 2. Edge maps

For this section let $E_a^{pq} \Rightarrow H^{p+q}$ be any first quadrant spectral sequence. Fix $p, q \geq 0$, $r \geq a$. Define ${}^z E_r^{pq}$ to be the elements of $E_r^{pq}$ which survive to $E_\infty^{pq}$; in the case of a filtered complex, this is just $Z_\infty^{pq}/B_r^{pq}$. We obtain the following sequence of maps:

$$E_r^{pq} \hookleftarrow {}^z E_r^{pq} \twoheadrightarrow E_\infty^{pq} \leftrightarrow F^p H^{p+q}/F^{p+1}H^{p+q} \hookrightarrow H^{p+q}/F^{p+1}H^{p+q} \twoheadleftarrow H^{p+q}. \tag{2.1}$$

We will say that this spectral sequence has an *$x$-axis edge map at $E_r^{pq}$* if ${}^z E_r^{pq} = E_r^{pq}$ and $F^{p+1}H^{p+q} = 0$. (Often we will omit the "$x$-axis" if it is clear from context.) Under these hypotheses (2.1) becomes

$$E_r^{pq} \twoheadrightarrow E_\infty^{pq} \leftrightarrow F^p H^{p+q} \hookrightarrow H^{p+q};$$

that is, we obtain a map

$$E_r^{pq} \to H^{p+q},$$

which explains the terminology. Note that if there is an $x$-axis edge map at $E_r^{pq}$ then there is an $x$-axis edge map at $E_{r'}^{pq}$ for all $r' \geq r$.

EXAMPLE 2.1. Suppose that for some $p, q, r$ the spectral sequence satisfies

$$E_r^{p+1,q-1} = E_r^{p+2,q-2} = \cdots = E_r^{p+q,0} = 0 \tag{2.2}$$

and

$$E_r^{p+r,q-r+1} = E_r^{p+r+1,q-r} = E_r^{p+r+2,q-r-1} = \cdots = E_r^{p+q+1,0} = 0. \tag{2.3}$$

(2.2) implies that the corresponding $E_\infty$ terms vanish; this in turn implies that

$$F^{p+1}H^{p+q}/F^{p+2}H^{p+q} = \cdots = F^{p+q}H^{p+q}/F^{p+q+1}H^{p+q} = 0;$$

thus $F^{p+1}H^{p+q} = 0$. (2.3) implies that $E_r^{pq} = {}^z E_r^{pq}$ since every later differential from the $(p, q)$ entry maps to 0. We conclude that under these conditions there is an $x$-axis edge map at $E_r^{pq}$.

There is a similar theory for $y$-axis edge maps: define ${}^b E_r^{pq}$ to be the quotient of $E_r^{pq}$ by all boundaries which ever map to it; in the case of a filtered complex, it is $Z_r^{pq}/B_\infty^{pq}$. There is a sequence of maps

$$H^{p+q} \hookleftarrow F^p H^{p+q} \twoheadrightarrow F^p H^{p+q}/F^{p+1}H^{p+q} \leftrightarrow E_\infty^{pq} \hookrightarrow {}^b E_r^{pq} \twoheadleftarrow E_r^{pq}. \tag{2.4}$$

We will say that a spectral sequence has a *$y$-axis edge map at $E_r^{pq}$* if ${}^b E_r^{pq} = E_r^{pq}$ and $F^p H^{p+q} = H^{p+q}$. Under these conditions (2.4) reduces to

$$H^{p+q} \twoheadrightarrow F^p H^{p+q}/F^{p+1}H^{p+q} \leftrightarrow E_\infty^{pq} \hookrightarrow E_r^{pq},$$

so we obtain a map

$$H^{p+q} \to E_r^{pq}.$$

Again, if there is a $y$-axis edge map at $E_r^{pq}$, then there is a $y$-axis edge map at $E_{r'}^{pq}$ for all $r' \geq r$.

EXAMPLE 2.2. Suppose that for some $p, q, r$ we have

$$E_r^{p-1,q+1} = E_r^{p-2,q+2} = \cdots = E_r^{0,p+q} = 0$$

and

$$E_r^{p-r,q+r-1} = E_r^{p-r-1,q+r} = E_r^{p-r-2,q+r+1} = \cdots = E_r^{0,p+q+1} = 0.$$

Then as in Example 2.1 one shows that there is a $y$-axis edge map at $E_r^{pq}$.

In the following we will work exclusively with $x$-axis edge maps, but the proofs all adapt immediately to the $y$-axis case as well.

## 3. Edge maps in spectral sequences of filtered complexes

Let $C_1^\bullet$ and $C_2^\bullet$ be filtered complexes as before. Suppose that we are given a map $C_1^\bullet \to C_2^\bullet$ compatible with the filtrations. This induces a map

$$E_r^{pq}(C_1^\bullet) \to E_r^{pq}(C_2^\bullet)$$

of spectral sequences and a map

$$H^{p+q}(C_1^\bullet) \to H^{p+q}(C_2^\bullet)$$

of cohomology groups.

PROPOSITION 3.1. *Suppose that for some $p, q, r$ there are edge maps*

$$E_r^{pq}(C_1^\bullet) \to H^{p+q}(C_1^\bullet);$$
$$E_r^{pq}(C_2^\bullet) \to H^{p+q}(C_2^\bullet).$$

*Then the diagram*

$$\begin{array}{ccc} E_r^{pq}(C_1^\bullet) & \longrightarrow & E_r^{pq}(C_2^\bullet) \\ \downarrow & & \downarrow \\ H^{p+q}(C_1^\bullet) & \longrightarrow & H^{p+q}(C_2^\bullet) \end{array}$$

*commutes.*

PROOF. Consider the expanded diagram

$$\begin{array}{ccc} E_r^{pq}(C_1^\bullet) & \longrightarrow & E_r^{pq}(C_2^\bullet) \\ \downarrow & & \downarrow \\ E_\infty^{pq}(C_1^\bullet) & \longrightarrow & E_\infty^{pq}(C_2^\bullet) \\ \downarrow & & \downarrow \\ F^pH^{p+q}(C_1^\bullet)/F^{p+1}H^{p+q}(C_1^\bullet) & \longrightarrow & F^pH^{p+q}(C_2^\bullet)/F^{p+1}H^{p+q}(C_2^\bullet) \\ \downarrow & & \downarrow \\ H^{p+q}(C_1^\bullet) & \longrightarrow & H^{p+q}(C_2^\bullet) \end{array} \tag{3.1}$$

All maps in (3.1) exist by our assumption that the edge maps exist. The first and last squares commute by the definitions of morphisms of spectral sequences and filtered complexes. This leaves the middle square: as we observed in (1.4), the maps

$$E_\infty^{pq}(C_i^\bullet) \to F^pH^{p+q}(C_i^\bullet)/F^{p+1}H^{p+q}(C_i^\bullet)$$

are induced by the inclusions $A_\infty^{pq} \hookrightarrow F^pC^{p+q}$, and now the commutativity of the middle square is clear as well. □

## 4. Edge maps in Grothendieck spectral sequences I

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left exact functors. Suppose that $G$ maps injective objects to $F$-acyclic objects. Under these hypotheses for each object $A$ of $\mathcal{A}$ one obtains a Grothendieck spectral sequence

$$E_2^{pq} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A). \tag{4.1}$$

(4.1) is constructed as follows: one begins with an injective resolution $A \to I^\bullet$ of $A$. The complex $G(I^\bullet)$ admits a Cartan-Eilenberg resolution $J^{\bullet\bullet}$. The Grothendieck spectral sequence (4.1) is the spectral sequence associated to the filtration by rows of the total complex of the double complex $F(J^{\bullet\bullet})$; see [**Wei94**, Section 5.8] for details.

Now let $A_1$ and $A_2$ be two objects of $\mathcal{A}$ and write $E_r^{pq}(A_1)$ and $E_r^{pq}(A_2)$ for the corresponding Grothendieck spectral sequences. Suppose that there is a morphism $A_1 \to A_2$ in $\mathcal{A}$. We now construct a corresponding morphism $E_r^{pq}(A_1) \to E_r^{pq}(A_2)$ of spectral sequences. Begin with injective resolutions $A_1 \to I_1^\bullet$ and $A_2 \to I_2^\bullet$. Standard properties of injective resolutions show that the map $A_1 \to A_2$ lifts to a map of complexes $I_1^\bullet \to I_2^\bullet$. This in turn yields a map of the complexes $G(I_1^\bullet) \to G(I_2^\bullet)$, and this lifts to a map $J_1^{\bullet\bullet} \to J_2^{\bullet\bullet}$ of any corresponding Cartan-Eilenberg resolutions. Taking $F$ of these complexes and passing to the associated total complex yields a map

$$\operatorname{Tot} F(J_1^{\bullet\bullet}) \to \operatorname{Tot} F(J_2^{\bullet\bullet}) \tag{4.2}$$

compatible with filtrations by rows or columns. (4.2) in turn yields a map of the associated Grothendieck spectral sequences. Proposition 3.1 can be restated in this situation as follows.

PROPOSITION 4.1. *Let $A_1 \to A_2$ be a morphism in $\mathcal{A}$. This morphism induces a morphism $E_r^{pq}(A_1) \to E_r^{pq}(A_2)$ of spectral sequences. Suppose further that for some $p, q, r$ there are edge maps*

$$E_r^{pq}(A_1) \to R^{p+q}(FG)(A_1);$$

$$E_r^{pq}(A_2) \to R^{p+q}(FG)(A_2).$$

*Then the diagram*

$$\begin{array}{ccc} E_r^{pq}(A_1) & \longrightarrow & E_r^{pq}(A_2) \\ \downarrow & & \downarrow \\ R^{p+q}(FG)(A_1) & \longrightarrow & R^{p+q}(FG)(A_2) \end{array}$$

*commutes.*

## 5. Edge maps in Grothendieck spectral sequences II

Suppose that one is given categories and functors forming a commutative diagram

$$\begin{array}{ccccc} \mathcal{A}_1 & \xrightarrow{G_1} & \mathcal{B}_1 & \xrightarrow{F_1} & \mathcal{C} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & \nearrow{\scriptstyle F_2} & \\ \mathcal{A}_2 & \xrightarrow{G_2} & \mathcal{B}_2 & & \end{array}$$

Suppose that the $F_i, G_i$ are left exact, that $G_i$ maps injectives to $F_i$-acyclics, and that $\alpha$ and $\beta$ are exact. Given an object $A$ of $\mathcal{A}_1$, these hypotheses allow us to form Grothendieck spectral sequences $E^{pq}_{r,1}$ for $A$ and $E^{pq}_{r,2}$ for $\alpha A$.

PROPOSITION 5.1. *Under the above hypotheses there is a natural map of spectral sequences $E^{pq}_{r,1} \to E^{pq}_{r,2}$ defined at the $r = 0$ stage. For $r = 2$ it agrees with the natural map*

$$R^pF_1R^qG_1(A) \to R^pF_2R^qG_2(\alpha A).$$

*Furthermore, if for some $p, q, r$ there exist edge maps*

$$E^{pq}_{r,1} \to R^{p+q}(F_1G_1)(A);$$

$$E^{pq}_{r,2} \to R^{p+q}(F_2G_2)(\alpha A);$$

*then the diagram*

$$\begin{array}{ccc} E^{pq}_{r,1} & \longrightarrow & E^{pq}_{r,2} \\ \downarrow & & \downarrow \\ R^{p+q}(F_1G_1)(A) & \longrightarrow & R^{p+q}(F_2G_2)(\alpha A) \end{array}$$

*commutes, where the bottom map is the natural map.*

PROOF. Begin with an injective resolution $A \to I_1^\bullet$. Let $J_1^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of the complex $G_1(I_1^\bullet)$. The Grothendieck spectral sequence for $F_1$ and $G_1$ is the spectral sequence associated to the filtration by rows of the total complex of $F_1(J_1^{\bullet\bullet})$.

The same construction for $\alpha A$ yields an injective resolution $I_2^\bullet$ and a Cartan-Eilenberg resolution $J_2^{\bullet\bullet}$ of $G_2(I_2^\bullet)$. Since $\alpha$ is exact, $\alpha I_1^\bullet$ is still exact; thus it is a resolution of $\alpha A$. This implies that there exists a map of complexes $\alpha I_1^\bullet \to I_2^\bullet$. This yields a map $G_2\alpha I_1^\bullet \to G_2 I_2^\bullet$. Since $G_2\alpha = \beta G_1$, we obtain a map $\beta G_1 I_1^\bullet \to G_2 I_2^\bullet$. $\beta$ is exact, so $\beta J_1^{\bullet\bullet}$ is a resolution of $\beta G_1 I_1^\bullet$ and thus maps to the Cartan-Eilenberg resolution $J_2^{\bullet\bullet}$.

Applying $F_2$ yields a map $F_2\beta J_1^{\bullet\bullet} \to F_2 J_2^{\bullet\bullet}$. Since $F_2\beta = F_1$, this is a map

$$F_1J_1^{\bullet\bullet} \to F_2J_2^{\bullet\bullet} \tag{5.1}$$

of double complexes. (5.1) induces a map of filtered complexes, and the proposition now follows from Proposition 3.1 so long as we check that the maps $E^{pq}_{2,1} \to E^{pq}_{2,2}$ and $R^{p+q}(F_1G_1)(A) \to R^{p+q}(F_2G_2)(\alpha A)$ are the natural maps.

We begin with $E^{pq}_{2,1} \to E^{pq}_{2,2}$. These maps are induced from (5.1) after first taking horizontal cohomology and then vertical cohomology. By the definition of a Cartan-Eilenberg resolution, horizontal cohomology of $J_i^{\bullet\bullet}$ yields resolutions (of complexes) $H^q(G_iI^\bullet) \to J_i^{q\bullet}$. By the definition of derived functors, we can identify these with resolutions

$$R^qG_1(A) \to J_1^{q\bullet}$$
$$R^qG_2(\alpha A) \to J_2^{q\bullet}.$$

Since the map $\beta J_1^{\bullet\bullet} \to J_2^{\bullet\bullet}$ sits over a map $\beta G_1 I_1^\bullet \to G_2 I_2^\bullet$, we now see immediately that the map $E_{1,1}^{pq} \to E_{1,2}^{pq}$ (which is what we obtain after horizontal cohomology) sits over the natural maps $\beta R^qG_1(A) \to R^qG_2(\alpha A)$. The $E_{1,i}^{pq}$'s form injective resolutions of these and the cohomology computes the right derived functors of the $F_i$. Since the map $E_{1,1}^{pq} \to E_{1,2}^{pq}$ sits over these natural maps we see now that $E_{2,1}^{pq} \to E_{2,2}^{pq}$ coincides with the natural maps $R^pF_1R^qG_1(A) \to R^pF_2R^qG_2(\alpha A)$, as claimed.

For the naturality of the other maps, it is immediate from the definitions that the map $\mathbf{R}^{p+q}F_1(G_1I_1^\bullet) \to \mathbf{R}^{p+q}F_2(G_2I_2^\bullet)$ induced by the spectral sequence map $E_{r,1}^{pq} \to E_{r,2}^{pq}$ is the natural map. We must check that the corresponding map $R^{p+q}(F_1G_1)(A) \to R^{p+q}(F_2G_2)(\alpha A)$ obtained from the collapsing of ${}^IE_{r,i}^{pq}$ is the natural map. Recall that to compute in this spectral sequence we first take vertical cohomology and then take horizontal cohomology. But since $G_i$ takes injectives to $F_i$-acyclics, after taking vertical cohomology (which computes $R^qF_i(G_iI^\bullet)$), we are left with a single row $R^0F_i(G_iI^\bullet) = F_iG_i(I^\bullet)$. One now sees as before that horizontal cohomology yields the usual map $R^{p+q}(F_1G_1)(A) \to R^{p+q}(F_2G_2)(\alpha A)$. □

## 6. Boundary maps of exact sequences of filtered complexes

Let $C^\bullet$ be a filtered complex as in Section 1. Note that in the first stage of the spectral sequence constructed from $C^\bullet$ the differentials are all horizontal. That is, for fixed $q$, $E_1^{\bullet q}$ can be considered as a complex as well, and its cohomology is nothing other than $E_2^{\bullet q}$.

Now let

$$0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$$

be an exact sequence of filtered complexes. Suppose in addition that for some $q$ the induced sequence

$$0 \to E_1^{\bullet q}(C_1^\bullet) \to E_1^{\bullet q}(C_2^\bullet) \to E_1^{\bullet q}(C_3^\bullet) \to 0$$

is also exact. We obtain long exact sequences of cohomology in both cases:

$$\cdots \to H^{n-1}(C_3^\bullet) \to H^n(C_1^\bullet) \to H^n(C_2^\bullet) \to H^n(C_3^\bullet) \to H^{n+1}(C_1^\bullet) \to \cdots \tag{6.1}$$

$$\cdots \to E_2^{p-1,q}(C_3^\bullet) \to E_2^{pq}(C_1^\bullet) \to E_2^{pq}(C_2^\bullet) \to E_2^{pq}(C_3^\bullet) \to E_2^{p+1,q}(C_1^\bullet) \to \cdots \tag{6.2}$$

In particular, we have boundary maps

$$H^{p+q}(C_3^\bullet) \to H^{p+q+1}(C_1^\bullet)$$
$$E_2^{pq}(C_3^\bullet) \to E_2^{p+1,q}(C_1^\bullet)$$

for each $p$.

Proposition 6.1. *Suppose that for some $p, q \geq 0$ there exist edge maps*

$$E_2^{pq}(C_3^\bullet) \to H^{p+q}(C_3^\bullet);$$
$$E_2^{p+1,q}(C_1^\bullet) \to H^{p+q+1}(C_1^\bullet).$$

*Then the diagram*

$$\begin{array}{ccc} E_2^{pq}(C_3^\bullet) & \longrightarrow & E_2^{p+1,q}(C_1^\bullet) \\ \downarrow & & \downarrow \\ H^{p+q}(C_3^\bullet) & \longrightarrow & H^{p+q+1}(C_1^\bullet) \end{array} \tag{6.3}$$

*commutes.*

Proof. Consider the diagram (in the notation of Section 1)

$$\begin{array}{ccccc} A_1^{p+1,q}(C_1^\bullet) & \longrightarrow & A_1^{p+1,q}(C_2^\bullet) & \longrightarrow & A_1^{p+1,q}(C_3^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ A_1^{pq}(C_1^\bullet) & \longrightarrow & A_1^{pq}(C_2^\bullet) & \longrightarrow & A_1^{pq}(C_3^\bullet) \end{array} \tag{6.4}$$

On the one hand, (6.4) surjects onto the diagram

$$\begin{array}{ccccc} E_1^{p+1,q}(C_1^\bullet) & \longrightarrow & E_1^{p+1,q}(C_2^\bullet) & \longrightarrow & E_1^{p+1,q}(C_3^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ E_1^{pq}(C_1^\bullet) & \longrightarrow & E_1^{pq}(C_2^\bullet) & \longrightarrow & E_1^{pq}(C_3^\bullet) \end{array}$$

from which the boundary maps of (6.2) are computed. We can therefore also compute these boundary maps after lifting to the diagram (6.4). (Recall that the boundary map is computed by lifting from $E_1^{pq}(C_3^\bullet)$ to $E_1^{pq}(C_2^\bullet)$, mapping to $E_1^{p+1,q}(C_2^\bullet)$ and pulling back to $E_1^{p+1,q}(C_1^\bullet)$.)

On the other hand, (6.4) naturally injects into the diagram

$$\begin{array}{ccccc} C^{p+q+1}(C_1^\bullet) & \longrightarrow & C^{p+q+1}(C_2^\bullet) & \longrightarrow & C^{p+q+1}(C_3^\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ C^{p+q}(C_1^\bullet) & \longrightarrow & C^{p+q}(C_2^\bullet) & \longrightarrow & C^{p+q}(C_3^\bullet) \end{array}$$

from which one computes the boundary maps of (6.1). The edge maps (when they exist) are induced by these injections. Since boundary maps are computed by the same procedure in both cases, it is now clear (6.3) commutes. □

## 7. Boundary maps of Grothendieck spectral sequences

We return now to the set-up of Section 4. We now assume that we have three left exact functors

$$F_1, F_2, F_3 : \mathcal{B} \to \mathcal{C}$$

and that $G : \mathcal{A} \to \mathcal{B}$ takes injectives to $F_i$-acyclic objects for each $i$. Suppose further that for any injective object $I$ of $\mathcal{B}$, the sequence

$$0 \to F_1(I) \to F_2(I) \to F_3(I) \to 0 \tag{7.1}$$

is exact.

Let us now go through the construction of the Grothendieck spectral sequence again. Begin with an object $A$ of $\mathcal{A}$. One first forms an injective resolution $A \to I^\bullet$ of $A$. Next, one takes a Cartan-Eilenberg resolution $J^{\bullet\bullet}$ of the complex $G(I^\bullet)$.

Applying each $F_i$ to this, we obtain three double complexes $F_i(J^{\bullet\bullet})$ in $\mathcal{C}$. In fact, we obtain an exact sequence

$$0 \to F_1(J^{\bullet\bullet}) \to F_2(J^{\bullet\bullet}) \to F_3(J^{\bullet\bullet}) \to 0$$

of double complexes by the exactness of (7.1). This in turn yields an exact sequence

$$0 \to \operatorname{Tot} F_1(J^{\bullet\bullet}) \to \operatorname{Tot} F_2(J^{\bullet\bullet}) \to \operatorname{Tot} F_3(J^{\bullet\bullet}) \to 0$$

of the total complexes, compatible with filtrations by rows and columns.

Consider now the filtrations of the $F_i(J^{\bullet\bullet})$ by rows. At the first stage of the associated spectral sequence one takes the horizontal cohomology. Since $J^{\bullet\bullet}$ is a Cartan-Eilenberg resolution, we can first form the cohomology of the complex $J^{\bullet\bullet}$ and then apply $F_i$. By the definition of a Cartan-Eilenberg resolution, the horizontal cohomology of $J^{\bullet\bullet}$ yields injective resolutions of the cohomology complex $H^\bullet(G(I^\bullet))$. The first stage of the spectral sequence is obtained by applying $F_i$ to this double complex; since the complex still consists of injectives, we again obtain an exact sequence of double complexes

$$0 \to E_{1,1}^{\bullet\bullet} \to E_{1,2}^{\bullet\bullet} \to E_{1,3}^{\bullet\bullet} \to 0$$

where the spectral sequence $E_{1,i}^{\bullet\bullet}$ is the Grothendieck spectral sequence for the composition of $F_i$ and $G$. In particular, we get boundary maps $E_2^{p+1,q} \to E_2^{p,q}$ as in (6.2). Given all of this, fixing $q$ and translating Proposition 6.1 into the language of Grothendieck spectral sequences yields the following result.

PROPOSITION 7.1. *Let $F_1, F_2, F_3, G$ be functors as above. Suppose that for some $p, q, r$ and $A$ there exist edge maps*

$$R^p F_3 R^q G(A) \to R^{p+q}(F_3 G)(A)$$

*and*

$$R^{p+1} F_1 R^q G(A) \to R^{p+q+1}(F_1 G)(A).$$

*Then the diagram*

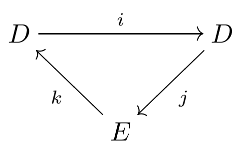
$$\begin{array}{ccc} R^p F_3 R^q G(A) & \longrightarrow & R^{p+1} F_1 R^q G(A) \\ \downarrow & & \downarrow \\ R^{p+q}(F_3 G)(A) & \longrightarrow & R^{p+q+1}(F_1 G)(A) \end{array}$$

*of edge maps and boundary maps commutes.*

## 8. Edge maps of exact couples

We conclude this appendix by considering the spectral sequence of an exact couple. Since [**Wei94**, Section 5.9] does not consider the cohomological case, we first set our notation. We will work in a somewhat restricted setting, as this is all which we will need for the applications.

We being by recalling the construction of the derived couple. Our terminology is adapted to our situation and is not standard. Consider an exact diagram $\mathcal{E}^1$



We define the (second) derived couple $\mathcal{E}^2$ to be

$$\begin{array}{ccc} i(D) & \xrightarrow{\quad i \quad} & i(D) \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j^{(2)}} \\ & \ker(jk)/\operatorname{im}(jk) & \end{array}$$

where $j^{(2)}(i(d)) = j(d)$. One checks easily that these maps are all well defined and that $\mathcal{E}^2$ is an exact couple; see [**Wei94**, Definition 5.9.1]. The $r^{\text{th}}$ derived couple $\mathcal{E}^r$ of $\mathcal{E}^1$ is defined to be the exact couple obtained from $\mathcal{E}^1$ after $r-1$ iterations of this construction.

Now suppose further that $D$ and $E$ are bigraded: $D = \oplus_{pq} D^{pq}$, $E = \oplus_{pq} E^{pq}$. Assume also that the maps $i$, $j$ and $k$ have bidegrees $(-1, 1)$, $(0, 0)$ and $(1, 0)$ respectively. Let $\mathcal{E}^r$ be the $r^{\text{th}}$ derived couple, and let $E_r^{pq}$ be the $(p, q)$-graded piece of the $E$-term in this couple. Letting $j^{(r)}k$ be the differential for $E_r^{pq}$, one sees immediately that $j^{(r)}k$ has bidegree $(r, -r+1)$, so that $E_r^{pq}$ is a spectral sequence.

We now make some more simplifying assumptions. Assume that $E$ is concentrated in the first quadrant; that $D^{pq} = 0$ for $q < p$; and that $i^{pq} : D^{pq} \to D^{p-1,q+1}$ is an isomorphism for $p \le 0$. In this situation one shows easily that $E_r^{pq}$ converges to $H^n = D^{0,n}$, with filtration $F^p H^n = i^r(D^{p,n-p})$ for any $r \ge p$ (here using the fact that $i$ is eventually an isomorphism). For any $r$ large enough so that $E_r^{pq} = E_\infty^{pq}$, the isomorphism

$$F^p H^n / F^{p+1} H^n = i^r(D^{p,n-p})/i^{r+1}(D^{p,n-p}) \cong E_r^{pq} = E_\infty^{pq} \tag{8.1}$$

is induced by $j^{(r)}$.

Suppose now that we have a morphism (of degree $(0, 0)$) $\mathcal{E}_1 \to \mathcal{E}_2$ of bigraded exact couples. It is immediate from the construction above that this induces a morphism

$$E_r^{pq}(\mathcal{E}_1) \to E_r^{pq}(\mathcal{E}_2)$$

of the corresponding spectral sequences. In the proposition below we consider only $x$-axis edge maps; the $y$-axis case is somewhat different.

Proposition 8.1. *Suppose that for some $p, q, r$ there exist $x$-axis edge maps*

$$E_r^{pq}(\mathcal{E}_1) \to H^{p+q}(\mathcal{E}_1);$$

$$E_r^{pq}(\mathcal{E}_2) \to H^{p+q}(\mathcal{E}_2).$$

*Then the diagram*

$$\begin{array}{ccc} E_r^{pq}(\mathcal{E}_1) & \longrightarrow & E_r^{pq}(\mathcal{E}_2) \\ \big\downarrow & & \big\downarrow \\ H^{p+q}(\mathcal{E}_1) & \longrightarrow & H^{p+q}(\mathcal{E}_2) \end{array}$$

*commutes, where the bottom map is induced from the map $D_1^{0,n} \to D_2^{0,n}$.*

Proof. Consider the diagram

$$\begin{array}{ccc}
E_r^{pq}(\mathcal{E}_1) & \longrightarrow & E_r^{pq}(\mathcal{E}_2) \\
\downarrow & & \downarrow \\
E_\infty^{pq}(\mathcal{E}_1) & \longrightarrow & E_\infty^{pq}(\mathcal{E}_2) \\
\downarrow & & \downarrow \\
F^p H^{p+q}(\mathcal{E}_1)/F^{p+1}H^{p+q}(\mathcal{E}_1) & \longrightarrow & F^p H^{p+q}(\mathcal{E}_2)/F^{p+1}H^{p+q}(\mathcal{E}_2) \\
\downarrow & & \downarrow \\
H^{p+q}(\mathcal{E}_1) & \longrightarrow & H^{p+q}(\mathcal{E}_2)
\end{array}$$

Recalling the definitions of each object, the only non-obvious commutativity is the middle square, and this is also clear since by (8.1) the maps can be taken to be induced by $j^{(r)}$ in both cases. □

APPENDIX B

# Gorenstein linear algebra

In this appendix we will prove the results on linear algebra over Gorenstein rings which we will need in the text. We also include a basic discussion of bilateral derivations and a few results on torsion $\mathbf{Z}_l$-modules.

## 1. Definitions

We will restrict ourselves to the case of finite, flat, local $\mathbf{Z}_l$-algebras.

DEFINITION 1.1. Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra with residue field $k$. We say that $A$ is *Gorenstein* if $\operatorname{Ext}^1_A(k, A) \cong k$.

Note that $\operatorname{Hom}_A(k, A) = 0$ since $A$ is torsion-free. Since $A$ necessarily has Krull dimension 1, this implies that this definition is the same (at least for finite, flat, local $\mathbf{Z}_l$-algebras) as that of [**Mat86**, Section 18].

It is a general fact [**Mat86**, Theorem 21.3] that local complete intersection rings (and therefore regular local rings) are Gorenstein. In particular, any ring of the form $\mathbf{Z}_l[x]/f(x)$ for a monic polynomial $f(x)$ is Gorenstein. Gorenstein rings are also necessarily Cohen-Macaulay; see [**Mat86**, Theorem 18.1].

The following characterization of finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebras is much more concrete and will be especially useful for us. For a finite, flat, local $\mathbf{Z}_l$-algebra $A$, we make $\operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ an $A$-module via $af(x) = f(ax)$ for $a \in A$ and $f \in \operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$. Note that $\operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ is isomorphic to $A$ as a $\mathbf{Z}_l$-module, but they need not be isomorphic as $A$-modules.

LEMMA 1.2. *Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra with maximal ideal $\mathfrak{m}$ and residue field $k$. Then the following conditions are equivalent.*

(1) *$A$ is Gorenstein;*
(2) $\dim_k (A/lA)[\mathfrak{m}] = 1$*;*
(3) $\operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ *is free of rank* 1 *as an $A$-module.*

PROOF. To prove the equivalence of the first two statements, consider the exact sequence

$$0 \to A \xrightarrow{l} A \to A/lA \to 0. \tag{1.1}$$

Since $\operatorname{Hom}_A(k, A) = 0$, we obtain from (1.1) an exact sequence

$$0 \to \operatorname{Hom}_A(k, A/lA) \to \operatorname{Ext}^1_A(k, A) \xrightarrow{l} \operatorname{Ext}^1_A(k, A). \tag{1.2}$$

But $l$ kills $k$ and thus kills $\operatorname{Ext}^1_A(k, A)$; therefore (1.2) yields an isomorphism

$$\operatorname{Hom}_A(k, A/lA) \cong \operatorname{Ext}^1_A(k, A).$$

$\operatorname{Hom}_A(k, A/lA)$ naturally identifies with $(A/lA)[\mathfrak{m}]$, from which the equivalence of (1) and (2) is clear.

To prove the equivalence of the second two statements, we will first show that $\dim_k(A/lA)[\mathfrak{m}] = 1$ is equivalent to $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l)$ being free of rank 1 over $A/lA$. First assume that $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l)$ is free of rank 1 over $A/lA$. We have

$$(A/lA)[\mathfrak{m}] \cong \mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l)[\mathfrak{m}] \cong \mathrm{Hom}_{\mathbf{F}_l}(A/\mathfrak{m}, \mathbf{F}_l). \tag{1.3}$$

The last term in (1.3) is easily seen to be a $k$-vector space of dimension 1, so that $(A/lA)[\mathfrak{m}]$ is as well, as claimed.

Next assume that $\dim_k(A/lA)[\mathfrak{m}] = 1$. We have an isomorphism

$$\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l) \otimes_{A/lA} A/\mathfrak{m} \cong \mathrm{Hom}_{\mathbf{F}_l}\big((A/lA)[\mathfrak{m}], \mathbf{F}_l\big). \tag{1.4}$$

$A/lA[\mathfrak{m}]$ is a one-dimensional $k$-vector space, so $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l) \otimes_{A/lA} k$ is as well. Nakayama's lemma shows that any lift of a generator of this module to $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l)$ will generate it as well; that is, $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l)$ is a cyclic $A/lA$-module. An easy $\mathbf{F}_l$-dimension count together with (1.4) now shows that it must be free of rank 1 over $A/lA$. This establishes the asserted equivalence.

To prove the lemma it now suffices to show that $\mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l) \cong A/lA$ is equivalent to $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \cong A$. For this, note that

$$\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \otimes_A A/lA \cong \mathrm{Hom}_{\mathbf{F}_l}(A/lA, \mathbf{F}_l).$$

From here a Nakayama's lemma argument and a $\mathbf{Z}_l$-rank count finish the proof. □

Note that Lemma 1.2 says that Gorenstein $\mathbf{Z}_l$-algebras are in some sense those which are self-dual. To the best of my knowledge, this sort of result first appeared in [**Maz77**, Chapter 2, Section 15].

## 2. Gorenstein traces and congruence elements

For the next two sections we fix a finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra $A$. We will call any $A$-generator of $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ a *Gorenstein trace* for $A$. Note that the choice of a Gorenstein trace is determined up to multiplication by an element of $A^\times$.

Let $\mathrm{tr} : A \to \mathbf{Z}_l$ be a choice of Gorenstein trace for $A$. Consider the ring $A \otimes_{\mathbf{Z}_l} A$, which we will always regard as an $A$-algebra via multiplication on the right factor of $A$. We have an isomorphism

$$\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} A \cong \mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A),$$

so that this module is free of rank 1 over $A \otimes_{\mathbf{Z}_l} A$. (In the $A \otimes_{\mathbf{Z}_l} A$-action on $\mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A)$, the left factor of $A$ must act on the domain, but the right factor can act on either the domain or the range by $A$-linearity.) tr induces an $A \otimes_{\mathbf{Z}_l} A$-generator $\mathrm{Tr} = \mathrm{tr} \otimes 1 : A \otimes_{\mathbf{Z}_l} A \to A$ of $\mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A)$:

$$\mathrm{Tr}(a \otimes a') = \mathrm{tr}(a)a'.$$

Let $\Delta : A \otimes_{\mathbf{Z}_l} A \to A$ be the diagonal map. Since Tr is a generator of $\mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A)$, we can write $\Delta = \iota\,\mathrm{Tr}$ for a unique $\iota \in A \otimes_{\mathbf{Z}_l} A$. We define the *congruence element* $\eta_{\mathrm{tr}}$ associated to tr to be $\Delta(\iota)$. That is, $\eta_{\mathrm{tr}}$ is the image of $1 \in A$ under the maps

$$A \cong \mathrm{Hom}_A(A, A) \xrightarrow{\circ\Delta} \mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A) \overset{\alpha\,\mathrm{Tr} \mapsto \alpha}{\longrightarrow} A \otimes_{\mathbf{Z}_l} A \xrightarrow{\Delta} A. \tag{2.1}$$

We define the *congruence ideal* of $A$ to be the $A$-ideal $\eta_{\mathrm{tr}}A$; the next lemma shows that this ideal is independent of the choice of Gorenstein trace tr.

LEMMA 2.1. *For any* $u \in A^\times$,

$$\eta_{u\,\mathrm{tr}} = u^{-1}\eta_{\mathrm{tr}}.$$

PROOF. $u\,\mathrm{tr}$ is also a generator of $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$, given by $u\,\mathrm{tr}(a) = \mathrm{tr}(ua)$. The associated generator of $\mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A)$ is therefore $\mathrm{Tr}' = (u \otimes 1)\,\mathrm{Tr}$. Writing $\Delta = \iota\,\mathrm{Tr}$, we have $\Delta = \iota(u^{-1} \otimes 1)\,\mathrm{Tr}'$. Thus $\eta_{u\,\mathrm{tr}} = \Delta(\iota(u^{-1} \otimes 1)) = u^{-1}\eta_{\mathrm{tr}}$ as claimed. □

For a definition of the congruence ideal for more general rings in a relative setting, see [**Len97**]. For us it will be critical to pin down the congruence element associated to a given Gorenstein trace, which more general formulations can not do.

The next result is useful for giving conditions under which there is an exact sequence $0 \to A \to A \to A/\eta A \to 0$.

LEMMA 2.2. *Let* $A$ *be a finite, flat, local, Gorenstein* $\mathbf{Z}_l$*-algebra and let* $\eta$ *be any congruence element for* $A$*. Then* $A$ *is reduced if and only if* $\eta$ *is a non-zero-divisor.*

## 3. Gorenstein duality

A key property of modules over Gorenstein rings is that it is possible to go between $A$-linear maps to $A$ and $\mathbf{Z}_l$-linear maps to $\mathbf{Z}_l$, in the sense of the following lemmas.

LEMMA 3.1. *Let* $M$ *be a finitely generated free* $A$*-module. Fix a Gorenstein trace* tr *of* $A$*. Then the map* $f \mapsto \mathrm{tr} \circ f$ *is an isomorphism*

$$\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l).$$

PROOF. One immediately reduces to the case where $M$ is free of rank 1, in which case this is just the definition of Gorenstein trace. □

LEMMA 3.2. *Let* $M$ *be a finitely generated* $A$*-module. Fix a Gorenstein trace* tr *of* $A$*. Then the map* $f \mapsto \mathrm{tr} \circ f$ *is an isomorphism*

$$\mathrm{Hom}_A(M, A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l) \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Q}_l/\mathbf{Z}_l).$$

PROOF. This follows from Lemma 3.1 on taking a resolution of $M$ by free $A$-modules. □

Let $M$ be a finitely generated free $A$-module. By Lemma 3.1 we can associate to a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ an isomorphism

$$\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l). \tag{3.1}$$

We can use (3.1) to define a map

$$h_{\mathrm{tr}} : \mathrm{End}_{\mathbf{Z}_l} M \to \mathrm{End}_A M$$

as the composition

$$\mathrm{End}_{\mathbf{Z}_l} M \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} M \cong \mathrm{Hom}_A(M, A) \otimes_{\mathbf{Z}_l} M \twoheadrightarrow \mathrm{Hom}_A(M, A) \otimes_A M \cong \mathrm{End}_A M.$$

$h_{\mathrm{tr}}$ is a fairly remarkable map, as it associates to a $\mathbf{Z}_l$-linear endomorphism of $M$ an $A$-linear endomorphism of $M$. The description of $h_{\mathrm{tr}}$ on $A$-linear endomorphisms of $A$ will be of fundamental importance to us.

LEMMA 3.3. *Let $M$ be a finitely generated free $A$-module. Let* tr *be a Gorenstein trace for $A$. Then the endomorphism*

$$\operatorname{End}_A M \to \operatorname{End}_{\mathbf{Z}_l} M \xrightarrow{h_{\mathrm{tr}}} \operatorname{End}_A M$$

*of* $\operatorname{End}_A M$ *is multiplication by* $\eta_{\mathrm{tr}}$.

PROOF. As usual, we reduce immediately to the case $M = A$. Tracing through the maps above, we see that we must prove that the identity map in $\operatorname{End}_{\mathbf{Z}_l} A$ maps to $\iota$ under the isomorphisms

$$\operatorname{End}_{\mathbf{Z}_l} A \cong \operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} A \cong A \otimes_{\mathbf{Z}_l} A, \tag{3.2}$$

where $\iota \in A \otimes_{\mathbf{Z}_l} A$ is such that $\Delta = \iota \operatorname{Tr}$. But this is clear after making the identification

$$\operatorname{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} A \cong \operatorname{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A) \tag{3.3}$$

and observing that the image of the identity map in (3.3) under (3.2) is precisely $\Delta$. □

## 4. Gorenstein pairings

The next result again goes back to [**Maz77**, Chapter 2, Section 15]; it is yet another relationship between Gorenstein rings and duality. Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra and let $M$ and $N$ be finitely generated $A$-modules. We say that a pairing

$$\psi : M \otimes_{\mathbf{Z}_l} N \to \mathbf{Z}_l$$

is *$A$-hermitian* if $\psi(am, n) = \psi(m, an)$ for all $a \in A$, $m \in M$, $n \in N$. Equivalently, this means that the natural maps

$$M \to \operatorname{Hom}_{\mathbf{Z}_l}(N, \mathbf{Z}_l)$$
$$N \to \operatorname{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l)$$

are maps of $A$-modules.

LEMMA 4.1. *Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra and let $H$ be a free $A$-module of rank 2. Then $A$ is Gorenstein if and only if there exists an $A$-hermitian perfect pairing*

$$\psi : H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l.$$

PROOF. First assume that such a $\psi$ exists. Since $\psi$ is perfect, there is an induced isomorphism

$$H \cong \operatorname{Hom}_{\mathbf{Z}_l}(H, \mathbf{Z}_l)$$

of $A$-modules. Let $A^\dagger$ denote the $A$-module $\operatorname{Hom}(A, \mathbf{Z}_l)$; we need to show that $A^\dagger$ is a free $A$-module of rank 1. But this follows from the isomorphism

$$A^\dagger \oplus A^\dagger \cong \operatorname{Hom}_{\mathbf{Z}_l}(H, \mathbf{Z}_l) \cong H \cong A \oplus A$$

and the fact that $A$ is flat over $\mathbf{Z}_l$.

For the other direction, let $\operatorname{tr} : A \to \mathbf{Z}_l$ be a Gorenstein trace and let $x, y$ be a basis for $H$. One checks easily that the pairing $\psi$ given by $\psi(ax + by, cx + dy) = \operatorname{tr}(ad - bc)$ is perfect, $A$-hermitian and even skew-symmetric. This completes the proof. □

We will call a pairing as in Lemma 4.1 a *Gorenstein pairing.*

The next result give a method of factoring Gorenstein pairings.

LEMMA 4.2. *Let $A$ be a finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra and let*

$$\psi : H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l$$

*be a Gorenstein pairing. Let* $\mathrm{tr} : A \to \mathbf{Z}_l$ *be any Gorenstein trace for $A$. Then the pairing $\psi$ factors as*

$$H \otimes_{\mathbf{Z}_l} H \twoheadrightarrow H \otimes_A H \xrightarrow{\psi'} A \xrightarrow{\mathrm{tr}} \mathbf{Z}_l$$

*where $\psi'$ is an $A$-linear perfect pairing.*

PROOF. Since $\psi$ is $A$-hermitian, it factors through some pairing $H \otimes_A H \to \mathbf{Z}_l$. Lemma 3.1 shows that this pairing factors through an $A$-linear pairing $\psi' : H \otimes_A H \to A$, and one checks immediately that $\psi'$ is perfect. □

## 5. Skew-symmetric Gorenstein pairings

As the proof of Lemma 4.1 indicates, the most natural Gorenstein pairings are those that are skew-symmetric. In this section we investigate the linear algebra associated to such a pairing.

Let $A$ be a finite, flat, local $\mathbf{Z}_l$-algebra and let $H$ be a free $A$-module of rank 2. Let

$$\psi : H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l$$

be a skew-symmetric Gorenstein pairing. $\psi$ induces a map

$$\wedge^2_A H \to \mathbf{Z}_l, \tag{5.1}$$

and it is easily checked that any choice of isomorphism of $A$ with $\wedge^2_A H$ turns (5.1) into a Gorenstein trace. Equivalently, we have the following lemma.

LEMMA 5.1. *Let $A$ be a finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra and let*

$$\psi : H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l$$

*be a skew-symmetric Gorenstein pairing. Let $x, y$ be a fixed $A$-basis of $H$ such that $\psi(x, y) = 1$. Then the map* $\mathrm{tr} : A \to \mathbf{Z}_l$ *given by* $\mathrm{tr}(a) = \psi(ax, y)$ *is a Gorenstein trace. Furthermore, $\psi$ factors as*

$$H \otimes_{\mathbf{Z}_l} H \twoheadrightarrow H \otimes_A H \xrightarrow{\psi'} A \xrightarrow{\mathrm{tr}} \mathbf{Z}_l$$

*where $\psi' : H \otimes_A H \to A$ is the pairing $\psi'(ax + by, cx + dy) = ad - bc$.*

PROOF. That the map tr is a Gorenstein trace is just the preceding argument made explicit. The factorization then follows by an easy computation. □

Recall that there are canonical decompositions

$$\mathrm{End}_A H \cong A \oplus \mathrm{End}^0_A H \tag{5.2}$$

$$H \otimes_A H \cong \wedge^2_A \oplus \mathrm{Sym}^2_A H. \tag{5.3}$$

LEMMA 5.2. *The map*

$$\mathrm{End}_A H \to \mathrm{Hom}_{\mathbf{Z}_l}(H \otimes_A H, \mathbf{Z}_l) \tag{5.4}$$

*sending $f : H \to H$ to the map $g_f(h \otimes h') = \psi(h \otimes f(h'))$ is an isomorphism of $A$-modules. Restricting to the direct summands of (5.2) and (5.3) induces isomorphisms*

$$A \cong \mathrm{Hom}_{\mathbf{Z}_l}(\wedge^2_A H, \mathbf{Z}_l) \tag{5.5}$$

$$\operatorname{End}^0_A H \cong \operatorname{Hom}_{\mathbf{Z}_l}(\operatorname{Sym}^2_A H, \mathbf{Z}_l). \tag{5.6}$$

PROOF. Fix a basis $x, y$ of $H$ such that $\psi(x, y) = 1$. Let tr be the associated Gorenstein trace as in Lemma 4.2, so that $\psi(ax+by, cx+dy) = \operatorname{tr}(ad-bc)$. Now let $f$ be an endomorphism of $H$ with matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with respect to $x, y$. The associated homomorphism

$$g_f : H \otimes_{\mathbf{Z}_l} H \to A$$

is given by

$$g_f(\alpha x + \beta y, \gamma x + \delta y) = \operatorname{tr}(\alpha\gamma c + \alpha\delta d - \beta\gamma a - \beta\delta b). \tag{5.7}$$

That (5.4) is an isomorphism follows from (5.7) by an easy calculation using a basis for $\operatorname{End}_A H$. The isomorphisms (5.5) and (5.6) also follow immediately on considering the cases $a = -d$ and $a = d, b = c = 0$. □

COROLLARY 5.3. *The isomorphism of Lemma 5.2 induces an isomorphism*

$$\operatorname{End}_A H \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \cong \operatorname{Hom}_{\mathbf{Z}_l}(H \otimes_A H, \mathbf{Q}_l/\mathbf{Z}_l).$$

## 6. Bilateral derivations

In this section we give the basic theory of bilateral derivations as developed in [**Maz**]. We make no effort to work in any level of generality which we will not need for our applications. Let $\mathcal{A}$ be a commutative $\mathbf{Z}_l$-algebra and let $M$ be an $\mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$-module. A *bilateral derivation* from $\mathcal{A}$ to $M$ is a $\mathbf{Z}_l$-linear map

$$\mathcal{D} : \mathcal{A} \to M$$

such that

$$\mathcal{D}(\beta\alpha) = (\alpha \otimes 1)\mathcal{D}(\beta) + (1 \otimes \beta)\mathcal{D}(\alpha)$$

for all $\alpha, \beta \in \mathcal{A}$. Note that if the action on $M$ factors through the diagonal map $\Delta : \mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A} \to \mathcal{A}$, then a bilateral derivation is nothing more than a derivation in the usual sense.

The fundamental example of a bilateral derivation is the map $\delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$ given by $\delta(\alpha) = \alpha \otimes 1 - 1 \otimes \alpha$. Note that the image of $\delta$ lies in the kernel $I$ of $\Delta$; one can show that $\delta : \mathcal{A} \to I$ is the universal bilateral derivation.

If $M$ is an $\mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$-module, define $M_\delta$ by

$$M_\delta = \{ m \in M \mid \delta(\alpha) m = 0 \text{ for all } \alpha \in \mathcal{A} \}.$$

$M_\delta$ is canonically an $\mathcal{A}$-module via $\Delta$.

LEMMA 6.1. *Let $\mathcal{D} : \mathcal{A} \to M$ be a bilateral derivation and let $\mathfrak{a}$ be an ideal of $\mathcal{A}$ such that $1 \otimes \mathfrak{a}$ and $\mathfrak{a} \otimes 1$ annihilate $M$. Then the restriction of $\mathcal{D}$ to $\mathfrak{a}$ yields an $\mathcal{A}$-module homomorphism*

$$\tilde{\mathcal{D}} : \mathfrak{a}/\mathfrak{a}^2 \to M_\delta.$$

PROOF. If $\beta \in \mathfrak{a}$, then the definition of a bilateral derivation shows that

$$(\alpha \otimes 1)\mathcal{D}(\beta) = \mathcal{D}(\beta\alpha) = \mathcal{D}(\alpha\beta) = (1 \otimes \alpha)\mathcal{D}(\beta) \tag{6.1}$$

for all $\alpha \in \mathcal{A}$. Thus $\mathcal{D}(\mathfrak{a}) \subseteq M_\delta$. If also $\alpha \in \mathfrak{a}$, then (6.1) shows that $\mathcal{D}(\alpha\beta) = 0$; thus $\mathcal{D}(\mathfrak{a}^2) = 0$. This proves the lemma. □

Now let $A$ be a finite, flat, local, reduced, Gorenstein $\mathbf{Z}_l$-algebra. Fix a Gorenstein trace tr for $A$ and let $\eta$ be the associated congruence element; $\eta$ is a non-zero divisor since $A$ is reduced. Let $M$ and $N$ be free $A$-modules of finite rank; $M \otimes_{\mathbf{Z}_l} N$ is an $A \otimes_{\mathbf{Z}_l} A$-module in the obvious way.

LEMMA 6.2. *There exists a unique $A$-module isomorphism*

$$\nu : (M \otimes_{\mathbf{Z}_l} N)_\delta \xrightarrow{\simeq} M \otimes_A N$$

*fitting into a commutative diagram*

$$\begin{array}{ccc} (M \otimes_{\mathbf{Z}_l} N)_\delta & \hookrightarrow & M \otimes_{\mathbf{Z}_l} N \\ \downarrow \nu & & \downarrow\!\!\downarrow \\ M \otimes_A N & \xhookrightarrow{\eta} & M \otimes_A N \end{array}$$

PROOF. The uniqueness of such a $\nu$ is clear. To define $\nu$ it suffices to consider the case $M = N = A$. Consider the sequence

$$(A \otimes_{\mathbf{Z}_l} A)_\delta \hookrightarrow A \otimes_{\mathbf{Z}_l} A \xrightarrow{\Delta} A. \tag{6.2}$$

Applying $\mathrm{Hom}_{\mathbf{Z}_l}(\cdot, \mathbf{Z}_l)$ to (6.2) yields a sequence

$$\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) \xrightarrow{\circ\Delta} \mathrm{Hom}_{\mathbf{Z}_l}(A \otimes_{\mathbf{Z}_l} A, \mathbf{Z}_l) \xrightarrow{f} \mathrm{Hom}_{\mathbf{Z}_l}\big((A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big). \tag{6.3}$$

The trace tr and Tr identify the first two terms with $A$ and $A \otimes_{\mathbf{Z}_l} A$, respectively. We claim that to define $\nu$ it suffices to prove that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}_l}(A \otimes_{\mathbf{Z}_l} A, \mathbf{Z}_l) & \xrightarrow{f} & \mathrm{Hom}_{\mathbf{Z}_l}\big((A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big) \\ \cdot\,\mathrm{Tr} \uparrow \simeq & & \simeq \uparrow \\ A \otimes_{\mathbf{Z}_l} A & \xrightarrow{\Delta} & A \end{array} \tag{6.4}$$

Indeed, given (6.4) it follows immediately from (2.1) that (6.3) fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbf{Z}_l}\big((A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big) \\ \cdot\,\mathrm{tr} \uparrow \simeq & & \simeq \uparrow \\ A & \xrightarrow{\eta} & A \end{array}$$

By duality we conclude that there is an isomorphism $\nu : (A \otimes_{\mathbf{Z}_l} A)_\delta \xrightarrow{\simeq} A$ such that

$$A \xleftarrow{\nu} (A \otimes_{\mathbf{Z}_l} A)_\delta \hookrightarrow A \otimes_{\mathbf{Z}_l} A \xrightarrow{\Delta} A$$

is multiplication by $\eta$, as claimed.

It thus suffices to construct (6.4). Note first that for any $\alpha \in A \otimes_{\mathbf{Z}_l} A$ such that $l\alpha \in (A \otimes_{\mathbf{Z}_l} A)_\delta$ we must have $\alpha \in (A \otimes_{\mathbf{Z}_l} A)_\delta$. It follows that $(A \otimes_{\mathbf{Z}_l} A)_\delta$ is a $\mathbf{Z}_l$-module direct summand of $A \otimes_{\mathbf{Z}_l} A$ and thus that

(6.5)

$$\mathrm{Hom}_{\mathbf{Z}_l}\big((A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big) \cong \mathrm{Hom}_{\mathbf{Z}_l}(A \otimes_{\mathbf{Z}_l} A, \mathbf{Z}_l) / \mathrm{Hom}_{\mathbf{Z}_l}\big(A \otimes_{\mathbf{Z}_l} A/(A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big).$$

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}_l}\big(A \otimes_{\mathbf{Z}_l} A/(A \otimes_{\mathbf{Z}_l} A)_\delta, \mathbf{Z}_l\big) & \hookrightarrow & \mathrm{Hom}_{\mathbf{Z}_l}(A \otimes_{\mathbf{Z}_l} A, \mathbf{Z}_l) \\ \cdot\,\mathrm{Tr} \uparrow \simeq & & \simeq \uparrow \cdot\,\mathrm{Tr} \\ \delta(A) & \hookrightarrow & A \otimes_{\mathbf{Z}_l} A \end{array} \tag{6.6}$$

by the definition of $\delta$. The desired diagram (6.4) now arises as the cokernel of (6.6) via (6.5). ☐

LEMMA 6.3. *Let $H$ be a free $A$-module of finite rank with an $A$-linear action of some group $G$. Assume also that every Jordan-Holder constituent of $H \otimes_{\mathbf{Z}_l} H$ has trivial $G$-invariants. Then there is a canonical isomorphism*

$$H^1(G, H \otimes_{\mathbf{Z}_l} H)_\delta \cong H^1\big(G, (H \otimes_{\mathbf{Z}_l} H)_\delta\big).$$

PROOF. This is an easy argument on the level of cocycles; we omit the details. ☐

Note that Lemma 6.3 applies in particular when $H \otimes_A k$ is absolutely irreducible of rank at least 2.

## 7. Torsion modules

In this section we collect some results on modules over finite, flat, local $\mathbf{Z}_l$-algebras. Let $A$ be a such a $\mathbf{Z}_l$-algebra, with maximal ideal $\mathfrak{m}$ and residue field $k$. If $T$ is an $A$-module and $\mathfrak{a}$ is an ideal of $A$, then we write $T[\mathfrak{a}]$ for the $\mathfrak{a}$-torsion in $T$:

$$T[\mathfrak{a}] = \big\{t \in T \mid \alpha t = 0 \text{ for all } \alpha \in \mathfrak{a}\big\}.$$

We will need the following simple facts about $A$-modules.

LEMMA 7.1. *Let $\alpha \in A$ be a non-zero divisor. Then $\alpha$ divides some power of $l$ in $A$ and $\alpha A$ has finite index in $A$.*

PROOF. Since $A$ is finite over $\mathbf{Z}_l$ and $\alpha$ is a non-zero divisor, $\alpha$ necessarily satisfies some monic linear equation of the form

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$$

with $a_i \in \mathbf{Z}_l$ and $a_0 \neq 0$. Thus $\alpha$ divides $a_0$, which proves the first statement. Since $a_0 A \subseteq \alpha A$ and $a_0 A$ visibly has finite index in $A$ (as $A$ is free of finite rank over $\mathbf{Z}_l$), the second statement is now clear as well. ☐

LEMMA 7.2. *Let $T$ be an $A$-module and let $t$ be a non-zero element of $T$ annihilated by some power of $l$. Then there exists $\alpha \in A$ such that $\alpha t \neq 0$ and $\alpha t \in T[\mathfrak{m}]$.*

PROOF. Since $t$ is $l$-power torsion, there is a largest power $l^n$ of $l$ such that $t_0 = l^n t \neq 0$; thus $t_0 \in T[l]$. Note that $T[l]$ is a module over the artinian ring $A/lA$. Let $\alpha_1, \dots, \alpha_m$ be generators of $\mathfrak{m}$ in $A$; they also generate the maximal ideal of $A/lA$ and are therefore nilpotent in this ring. It follows that some power of $\alpha_1$ annihilates $t_0$; let $n_1$ be the smallest such power, and set $t_1 = \alpha_1^{n_1-1} t_0$. Continuing in this way, we obtain a non-zero element $t_m = \alpha t$ where $\alpha = l^n \alpha_1^{n_1-1} \cdots \alpha_m^{n_m-1}$. This $t_m$ is killed by every generator of $\mathfrak{m}$, so it is clearly in $T[\mathfrak{m}]$. ☐

LEMMA 7.3. *Let $T$ be a $\mathbf{Z}_l$-torsion $A$-module. If $T[\mathfrak{m}] = 0$, then $T = 0$.*

PROOF. Suppose that $T \neq 0$ and let $t$ be a non-zero element of $T$. By Lemma 7.2 there is some $\alpha \in A$ such that $\alpha t$ is non-zero and annihilated by $\mathfrak{m}$. Thus $T[\mathfrak{m}] \neq 0$, which yields the desired contradiction. □

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