SELMER GROUPS AND CHOW GROUPS OF SELF-PRODUCTS OF ALGEBRAIC VARIETIES

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Abstract. Let $\mathfrak{X}$ be a proper flat scheme over the ring of integers of a global field. We show that the Tate conjecture and the finiteness of the Chow group of vertical cycles on self-products of $\mathfrak{X}$ implies the vanishing of the dual Selmer group of certain twists of tensor powers of representations occurring in the étale cohomology of $\mathfrak{X}$.

1. Introduction

Let $\mathfrak{X}$ be a proper flat scheme over the ring of integers $\mathcal{O}$ of a number field F and let ℓ be a prime such that $\mathfrak{X}$ has good reduction at every place of F dividing ℓ. (We also consider the case of a global function field F; see Section 4.1 for conventions in that case.) Let V be the contragredient of an irreducible Galois representation occurring in an étale cohomology group $H^m_{\text{ét}}(\mathfrak{X}_{\bar{F}}, \mathbf{Q}_\ell)$. Assume that the image of Galois acting on V is an open subgroup of either the general linear group of V or one of its maximal symplectic or orthogonal subgroups; in the first case set $\kappa = \dim V$ and in the latter cases set $\kappa = 2$. For any positive integer r with mr even let Γ_r denote the subgroup of vertical cycles on smooth fibers in the codimension $(\frac{mr}{2}+1)$ Chow group of the r-fold self-product $\mathfrak{X}^r$ of $\mathfrak{X}$ over $\mathcal{O}$. In this paper we prove the following theorem.

Theorem 1.1. *Let r be a positive multiple of κ such that mr is even. Assume that a strong form (Conjecture 4.1) of the Tate conjecture holds for closed fibers of $\mathfrak{X}^r$ and that Γ_r has finite exponent. Then the Selmer group $\mathcal{S}\big(F, V^{\otimes r}(-\frac{mr}{2})\big)$ vanishes.*

We should note that both the hypothesis and the conclusion of this theorem are predicted by the usual conjectures on special values of motivic L-functions. Indeed, one can view this theorem as showing that, for certain motives, one part of the conjectures of Beilinson and Bloch-Kato (the existence of motivic elements) implies another (the relation between the dimension of a Selmer group and the order of a vanishing of an L-function). See [10, Section 4.1] for a discussion. In addition, the results of this paper are in some sense complementary to those of [7] and [8], where results on Selmer groups are used to control torsion cycles on surfaces. Here, by contrast, we show that if enough vertical cycles are torsion, then one can control Selmer groups.

In many ways this theorem can be regarded as a massive generalization of part of Flach's finiteness theorem (see [3]) for Selmer groups of symmetric squares of elliptic curves. In our notation, Flach considers the case $F = \mathbf{Q}$, $\mathfrak{X}$ an elliptic curve, V its Tate space, and $r = 2$. He uses a modular unit construction to show that many

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cycles on $\mathfrak{X}^2$ are torsion; from these torsion cycles he constructs a geometric Euler system in the cohomology of $V^{\otimes 2}$. He then applies methods of Kolyvagin to show that the existence of this Euler system implies the vanishing of the Selmer group of $V^{\otimes 2}(-1) = \mathrm{Sym}^2 V(-1) \oplus \mathbf{Q}_\ell$.

In any attempt to generalize Flach's work two questions arise. First, what sort of algebraic cycles are required to produce a geometric Euler system? Second, what properties of the representation $V^{\otimes 2}(-1)$ allow one to use the geometric Euler system to control the Selmer group? The latter question is answered in [10]: a geometric Euler system can be used to control the Selmer group of a locally isotropic Galois representation. (An irreducible Galois representation is said to be *locally isotropic* if there is an open subset U of the absolute Galois group of F such that every $\gamma \in U$ fixes some non-zero vector (possibly depending on γ) in the representation space.)

There are thus two main tasks in proving our theorem. In Section 4 we show that, in the presence of the Tate conjecture, the finite exponent condition on Γ_r implies the existence of the algebraic cycles required to produce a geometric Euler system. It then remains to show that the Galois representation $V^{\otimes r}(-\frac{mr}{2})$ is locally isotropic. This is done in Section 3 using the study of locally isotropic representations of algebraic groups in Section 2. Unfortunately, verifying the hypotheses of our theorem in any specific case (the analogue of Flach's modular unit construction) appears quite difficult; we know of no applications beyond those to adjoint representations of classical and Drinfeld modular forms considered in [10, Section 5]. Nevertheless, we hope that this paper can offer an approach to controlling Selmer groups of Galois representations associated to more general automorphic forms in the event that some analogue of the modular unit construction is discovered.

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2. Locally isotropic representations of algebraic groups

2.1. **Generalities.** Let G be an algebraic group over an algebraically closed field K of characteristic zero. An irreducible representation $\rho : \mathrm{G} \to \mathrm{GL}_V$ of G on a finite dimensional K-vector space V is said to be *locally isotropic* if for each $\gamma \in \mathrm{G}(K)$ there is a non-zero $v \in V$ with $\rho(\gamma)v = v$. The following fact is immediate from elementary linear algebra.

Lemma 2.1. *Suppose that $\rho : \mathrm{G} \to \mathrm{GL}_V$ is the base change of a representation $\rho_0 : \mathrm{G}_0 \to \mathrm{GL}_{V_0}$ defined over some subfield K_0 of K. If every irreducible constituent of ρ is locally isotropic, then every $\gamma \in \mathrm{G}_0(K_0)$ fixes some non-zero vector in V_0.*

Fix a maximal torus T of G. Recall that a character $\chi : T \to \mathbf{G}_{\mathrm{m}}$ is said to be a *weight* of a representation $\rho : \mathrm{G} \to \mathrm{GL}_V$ if there is a non-zero vector $v \in V$ such that $\rho(\gamma)v = \chi(\gamma)v$ for all $\gamma \in T(K)$. As the image of T in GL_V is diagonalizable, V is necessarily the direct sum of its χ-eigenspaces as χ runs through the weights of ρ. The following lemma, the proof of which was communicated to me by Eric Sommers, gives a useful criterion for local isotropy.

Lemma 2.2. *If $\rho : \mathrm{G} \to \mathrm{GL}_V$ is an irreducible representation, then ρ is locally isotropic if and only if the trivial character is a weight of ρ.*

Proof. First, let ρ be locally isotropic and let $\chi_1, \ldots, \chi_r : T \to \mathbf{G}_{\mathrm{m}}$ be the weights of ρ. Since ρ is locally isotropic, every $\gamma \in T$ fixes some non-zero vector in V

and thus has a trivial eigenvalue. As the eigenvalues of γ acting on V are exactly $\chi_1(\gamma), \ldots, \chi_r(\gamma)$, one of these must be trivial, so that $T = \cup \ker \chi_i$. Each $\ker \chi_i$ is Zariski closed in T, so it follows that $T = \ker \chi_i$ for some i, as claimed.

For the converse, assume that ρ has a trivial weight and choose $\gamma \in \mathrm{G}(K)$. Let $\gamma = su$ be the Jordan decomposition of γ (so that s is semisimple, u is unipotent and $su = us$). Let $V' \subseteq V$ be the trivial eigenspace for $\rho(s)$; it is non-zero since ρ has a trivial weight and s is conjugate to some element of $T(K)$. The unipotent endomorphism $\rho(u)$ stabilizes V' (since s and u commute), so that it has a non-zero fixed vector $v \in V'$. Then $\rho(\gamma)v = v$, which shows that ρ is locally isotropic. □

2.2. **Semisimple groups.** We can use Lemma 2.2 to give a simple characterization of locally isotropic representations of semisimple groups. Let G be semisimple and fix a maximal torus T. Recall that the weight lattice Λ is the lattice of all weights (with respect to T) of all representations of G; the root lattice Λ_0 is the finite index sublattice generated by the weights occurring in the adjoint representation of G. For a representation ρ we write $w(\rho) \subseteq \Lambda$ for the set of weights of ρ.

Lemma 2.3. *An irreducible representation ρ of a semisimple group* G *is locally isotropic if and only if $w(\rho) \cap \Lambda_0$ is non-empty, in which case $w(\rho) \subseteq \Lambda_0$.*

Proof. Let $S \subseteq \Lambda$ denote the orbit of a highest weight of ρ under the Weyl group of G; it is the set of vertices of a convex polytope $\mathcal{P}$ centered at the origin. By the theory of semisimple groups (see [5, Chapter 14], for example) $w(\rho)$ is precisely $\mathcal{P} \cap (s + \Lambda_0)$ for any $s \in S$. As the trivial weight 0 always lies in $\mathcal{P}$, the lemma now follows from Lemma 2.2. □

2.3. **Reductive groups.** We switch now to the setting of reductive groups. Let G denote one of the groups GL_n, GSp_{2n}, GO_{2n}, GO_{2n+1} and let $\pi : \mathrm{G} \hookrightarrow \mathrm{GL}_N$ be the standard representation of G (so that $N = n, 2n, 2n, 2n+1$, respectively); in the former case we take π to be the identity map, while in the latter cases we normalize π so that G is the group of similitudes of the bilinear form given by

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. In each case the set of elements of G diagonal in this representation form a maximal torus T of G.

The coordinate characters $\chi_1, \ldots, \chi_N$ of T generate the weight lattice Λ of G, freely if $\mathrm{G} = \mathrm{GL}_n$ and subject to the relations $(1 \leq i, j \leq n)$

$$\chi_i + \chi_{n+i} = \chi_j + \chi_{n+j} \text{ for } \mathrm{G} = \mathrm{GSp}_{2n}, \mathrm{GO}_{2n},$$

$$\chi_i + \chi_{n+i} = \chi_j + \chi_{n+j} = 2\chi_{2n+1} \text{ for } \mathrm{G} = \mathrm{GO}_{2n+1}.$$

There is an exact sequence

$$1 \to \mathrm{G}' \longrightarrow \mathrm{G} \xrightarrow{\mu} \mathbf{G}_{\mathrm{m}} \to 1$$

with G' semisimple and μ the fundamental character of G. (This character is $\det \circ \pi$ in the general linear case and is the multiplier character in the remaining cases.) This induces an exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\delta} \Lambda \xrightarrow{\eta} \Lambda' \to 0$$

with Λ' the weight lattice of G' and

$$\delta(1) = \begin{cases} \chi_1 + \cdots + \chi_N & \mathrm{G} = \mathrm{GL}_N; \\ \chi_i + \chi_{n+i} & \mathrm{G} = \mathrm{GSp}_N, \mathrm{GO}_N. \end{cases}$$

We have

$$w(\pi) = \{\chi_1, \ldots, \chi_N\}; \qquad w(\mu) = \{\delta(1)\}. \tag{2.1}$$

Let $\varepsilon : \Lambda \to \mathbf{Z}$ be the map sending each χ_i to 1; set $\kappa = \kappa(\mathrm{G}) := \varepsilon\delta(1)$. By direct computation one sees that the kernel of the induced map $\varepsilon' : \Lambda' \to \mathbf{Z}/\kappa\mathbf{Z}$ contains the root lattice Λ'_0 of G'. (In fact, $\Lambda'_0 = \ker \varepsilon'$ except in the odd orthogonal case.)

Proposition 2.4. *Let ρ be an irreducible constituent of $\pi^{\otimes r} \otimes \mu^s$ for $r \geq 0$ and $s \in \mathbf{Z}$. Then ρ is locally isotropic if and only if $r = -\kappa s$.*

Proof. The weights of $\pi^{\otimes r}$ are r-fold sums of the elements of $w(\pi)$, so by (2.1) we have $\varepsilon(w(\rho)) = r + \kappa s$. If ρ is locally isotropic, then by definition 0 lies in $w(\rho)$, so we must have $r + \kappa s = 0$. Conversely, if $r + \kappa s = 0$, then $\varepsilon'\eta(w(\rho)) = 0$. Thus $\eta(w(\rho)) \subseteq \Lambda'_0$. But $\eta(w(\rho))$ is simply the set of weights of ρ regarded as a representation of G'. One sees easily that ρ is still irreducible as a representation of G', so that by Lemma 2.3 $\eta(w(\rho))$ contains 0. Thus there is a $t \in \mathbf{Z}$ such that $\delta(t) \in w(\rho)$. Since $\varepsilon\delta(t) = r + \kappa s = 0$, we must have $t = 0$, which completes the proof. □

Remark 2.5. For these groups every irreducible representation is a constituent of some $\pi^{\otimes r} \otimes \mu^s$, so this gives a complete classification of locally isotropic representations in these cases.

3. Galois representations

3.1. **Generalities.** Let F be a global field with absolute Galois group $\mathcal{G}_F$ and consider a finite dimensional $\mathbf{Q}_\ell$-vector space V (with $\ell \neq \operatorname{char} F$) with a continuous $\mathbf{Q}_\ell$-linear action of $\mathcal{G}_F$. We assume that V is irreducible, unramified at almost all places of F, and pure of some weight m (in the sense that for almost all places v, the eigenvalues of a geometric Frobenius element at v on V are algebraic with absolute value $\mathrm{Nm}(v)^{m/2}$ under any embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$). If F has characteristic zero, we further assume that V is crystalline at all places of F dividing ℓ.

Fix a maximal subfield K of the division algebra $\mathrm{End}_{\mathbf{Q}_\ell[\mathcal{G}_F]} V$. From now on we regard V as a K-vector space and write GL_V for the K-algebraic group of K-linear automorphisms of V. By [6, Proposition 15.7] we have $\mathrm{End}_{K[\mathcal{G}_F]} V = K$, so that V is absolutely irreducible as a representation over K.

3.2. **Galois groups.** Let $\mathcal{G}_F$ act on the graded K-algebra

$$\mathbf{E} = \bigoplus_{r \geq 0} \mathrm{End}_K V^{\otimes r}$$

via the adjoint action. For any K-algebra A, $\mathbf{E} \otimes_K A$ has a natural action of $\mathrm{GL}_V(A)$, and we let G_V denote the algebraic group over K with

$$\mathrm{G}_V(A) = \{\gamma \in \mathrm{GL}_V(A) : \gamma(e) = e \text{ for all } e \in (\mathbf{E} \otimes_K A)^{\mathcal{G}_F}\}.$$

Let $\pi : \mathrm{G}_V \hookrightarrow \mathrm{GL}_V$ be the obvious inclusion of algebraic groups. The image of $\mathcal{G}_F$ acting on V lies in $\mathrm{G}_V(K)$; let $\rho : \mathcal{G}_F \to \mathrm{G}_V(K)$ denote this map.

Set $N = \dim_K V$. We say that V is *standard over* K if $\rho(\mathcal{G}_F)$ is open in $\mathrm{G}_V(K)$ and π is isomorphic to the standard representation of GL_N, GSp_N or GO_N over $\bar{K}$. (In particular, $\mathrm{G}_V \times_K \bar{K}$ is then isomorphic to one of these groups.) It is conjectured that the openness condition holds if ρ is motivic; see [9] for details.

3.3. **Locally isotropic Galois representations.** We say that the Galois representation V is *locally isotropic* if there is an open subset $U \subseteq \mathcal{G}_F$ such that each $\gamma \in U$ fixes some non-zero vector in V. Proposition 2.4 allows one to deduce the existence of many locally isotropic Galois representations among twists of tensor powers of standard representations.

Lemma 3.1. *Let V be standard over K and let r be a positive multiple of $\kappa = \kappa(G_V)$ such that $mr = 2d$ is even. Then $V^{\otimes r}(d)$ is semisimple and every irreducible constituent is locally isotropic.*

Proof. Let μ be the fundamental character of G_V. Since V is pure of weight m, for almost all places v of F the eigenvalues of a Frobenius element Frob_v at v are algebraic with complex absolute values $\mathrm{Nm}(v)^{m/2}$. By the definition of the character μ, it follows that $\mu \circ \rho(\mathrm{Frob}_v) \in K^\times$ is algebraic with complex absolute values $\mathrm{Nm}(v)^{m\kappa/2}$. Thus $\mu^{-r/\kappa} \circ \rho(\mathrm{Frob}_v)$ is algebraic with complex absolute values $\mathrm{Nm}(v)^{(m\kappa/2)(-r/\kappa)} = \mathrm{Nm}(v)^{-d}$. As every continuous character $\mathcal{G}_F \to K^\times$ differs from a power of the cyclotomic character ε by a character of finite order and $\varepsilon(\mathrm{Frob}_v)$ has absolute values $\mathrm{Nm}(v)^{-1}$, we conclude that $\mu^{-r/\kappa} \circ \rho : \mathcal{G}_F \to K^\times$ equals $\varepsilon^d \chi$ with ε the cyclotomic character and χ of finite order.

It follows that $V^{\otimes r}(d) \otimes \chi$ is the realization of the Galois representation $(\pi^{\otimes r} \otimes \mu^{-r/\kappa}) \circ \rho$. By the theory of Schur representations (see [5]), the representation $\pi^{\otimes r} \otimes \mu^{-r/\kappa}$ is semisimple; as $\rho(\mathcal{G}_F)$ is open (and thus Zariski dense) in $\mathrm{G}_V(K)$, the Galois representation $V^{\otimes r}(d) \otimes \chi$ is then semisimple as well.

Let λ be an irreducible constituent of $\pi^{\otimes r} \otimes \mu^{-r/\kappa}$ (so that $\lambda \circ \rho$ is an irreducible constituent of $V^{\otimes r}(d) \otimes \chi$). By Proposition 2.4 and Lemma 2.1 $\lambda(\gamma)$ has a trivial eigenvalue for every $\gamma \in \mathrm{G}_V(K)$. Thus $\lambda \circ \rho(\sigma)$ has a trivial eigenvalue for every $\sigma \in \mathcal{G}_F$, so that the Galois representation $\lambda \circ \rho$ is locally isotropic in the sense defined above with $U = \mathcal{G}_F$.

We have seen that $V^{\otimes r}(d) \otimes \chi$ is semisimple and that every irreducible constituent is locally isotropic. As semisimplicity and local isotropy are invariant under twisting by characters of finite order, the lemma follows. ☐

3.4. **Selmer groups.** For every non-archimedean place v of F define

$$H^1_s(F_v, V) = \begin{cases} H^1(\mathcal{I}_v, V)^{\mathcal{G}_v} & v \nmid \ell; \\ \operatorname{im}\bigl(H^1(F_v, V) \to H^1(F_v, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})\bigr) & v \mid \ell; \end{cases}$$

with $\mathcal{G}_v$ (resp. $\mathcal{I}_v$) the local Galois group (resp. inertia group) at v and B_{cris} the ring of Fontaine. If v is archimedean we simply set $H^1_s(F_v, V) = H^1(F_v, V)$. For a set of places P define the *P-Selmer group* of V by

$$\mathcal{S}_P(F, V) = \ker\Bigl(H^1(F, V) \to \prod_{v \notin P} H^1(F_v, V) \to \prod_{v \notin P} H^1_s(F_v, V)\Bigr);$$

we simply write $\mathcal{S}(F, V)$ for $\mathcal{S}_\phi(F, V)$. For motivic V it is conjectured that $\mathcal{S}(F, V)$ is related to the order of the L-function of the dual of V at $s = 1$; see [4].

Remark 3.2. One can analogusly define $\mathcal{S}(F, V/T)$ for a Galois stable lattice T in V, and one sees easily that $\mathcal{S}(F, V/T)$ is finite if and only if $\mathcal{S}(F, V) = 0$.

Set $W = \mathrm{Hom}_K(V, K(1))$. We say that W admits a *geometric Euler system* if for some (or equivalently for any) Galois stable $\mathcal{O}_K$-lattice $T \hookrightarrow W$ there is a non-zero $e \in \mathcal{O}_K$ such that

$$\mathrm{Coker}\big(\mathcal{S}_{\{v\}}(F,T) \to H^1_s(F_v,T)\big)$$

is annihilated by e for almost all places v of F.

Proposition 3.3. *If V is locally isotropic and W admits a geometric Euler system, then $\mathcal{S}(F,V) = 0$.*

This is immediate from [10, Proposition 2.3] (with any choice of minimal element γ).

Remark 3.4. The proof of [10, Proposition 2.3] rests heavily on the fact that V is locally isotropic: the geometric Euler system is combined with Poitou-Tate duality to obtain bounds on the Selmer groups of finite quotients of a lattice in V. The local isotropy of V makes these bounds independent of the finite quotient, and taking limits yields the desired vanishing of $\mathcal{S}(F,V)$. We must emphasize that the assumption of local isotropy is crucial: without it one obtains destructive congruences in passing to finite quotients which causes the bounds on Selmer groups to grow too rapidly to yield a bound in the limit.

4. Chow groups

4.1. **Set-up.** Let F be a global field. If F is a number field, let $\mathcal{O}_F$ denote the ring of integers of F; if F is a function field, fix a place ∞ of F and let $\mathcal{O}_F$ denote the subring of F of elements with poles at most at ∞. Let $\mathfrak{X}$ be a proper flat generically smooth scheme over $\mathrm{Spec}\,\mathcal{O}_F$. Let X denote the generic fiber of $\mathfrak{X}$ and let S denote the open subscheme of $\mathrm{Spec}\,\mathcal{O}_F$ over which $\mathfrak{X}$ is smooth. Let ℓ be a rational prime such that $\mathrm{char}\,F \neq \ell$ and such that every place of F dividing ℓ lies in S. (If F is a function field, we regard this last condition as vacuous.)

Let W be an irreducible $\mathbf{Q}_\ell[\mathcal{G}_F]$-quotient of $H^m_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell)$ for some m (with $\bar{X}$ the base change of X to a separable closure of F). By work of Deligne and Faltings (see [1] and [2]) W and $V := \mathrm{Hom}_{\mathbf{Q}_\ell}(W, \mathbf{Q}_\ell)$ satisfy the hypotheses of Section 3.1 with weights m and $-m$, respectively. Fix a maximal subfield $K \hookrightarrow \mathrm{End}_{\mathbf{Q}_\ell[\mathcal{G}_F]} V$. From now on we regard V as an absolutely irreducible $K[\mathcal{G}_F]$-module. We also assume that V is standard over K in the sense of Section 3.2; set $N = \dim_K V$ and $\kappa = \kappa(\mathrm{G}_V)$.

Let r be a positive multiple of κ such that $mr = 2d$ is even. By the Künneth theorem $W^{\otimes r}$ can be realized as a quotient of $H^{mr}_{\text{ét}}(\bar{X}^r, \mathbf{Q}_\ell)$. Define a lattice $T_r \subseteq W^{\otimes r}$ as the $\mathcal{O}_K$-span of the image of $H^{mr}_{\text{ét}}(\bar{X}^r, \mathbf{Z}_\ell)$. We now state our a strong form of the Tate conjecture. For each place v define the space of local Tate cycles $\mathfrak{T}_v T_r = T_r(d)^{\mathcal{G}_v}$ and let $\mathrm{CH}^d X^r_v$ denote the Chow group of codimenson d cycles on X^r_v.

Conjecture 4.1. *The cokernel of the cycle class map*

$$c_v : \mathrm{CH}^d X^r_v \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{mr}_{\acute{e}t}\big(\bar{X}^r_v, \mathbf{Z}_\ell(d)\big)^{\mathcal{G}_v} \to \mathfrak{T}_v T_r$$

is bounded independent of $v \in S$ not dividing ℓ.

4.2. **The main result.** Let $\mathrm{CH}^{d+1}\mathfrak{X}^r$ denote the Chow group of the scheme $\mathfrak{X}^r$. For each place v there is a map

$$\iota_v : \mathrm{CH}^d X^r_v \to \mathrm{CH}^{d+1}\mathfrak{X}^r$$

given by regarding a cycle on X_v^r as a vertical cycle on $\mathfrak{X}^r$. Let Γ_r denote the subgroup of $\mathrm{CH}^{d+1}\mathfrak{X}^r$ spanned by the image of ι_v for all $v \in S$.

Define the higher Chow group $\mathrm{CH}^{d,1}X^r$ as the quotient of the group

$$\Big\{\textstyle\sum(Z_i, f_i); Z_i \hookrightarrow X \text{ codim } d,\ f_i \in k(Z_i)^\times : \sum \mathrm{div}_{Z_i} f_i = 0\Big\}$$

by the tame equivalence relation; see [10, Section 3.1] for precise definitions. We recall two facts about this group. First, for any place v there is a map

$$\mathrm{div}_v : \mathrm{CH}^{d,1}X^r \to \mathrm{CH}^d X_v^r$$

sending $\sum(Z_i, f_i)$ to $\sum \mathrm{div}_{Z_{i,v}} f_i$ with $Z_{i,v}$ the fiber at v of the closure of Z_i in $\mathfrak{X}^r$. Second, by [10, Proposition 3.1] there is a regulator map

$$\mathcal{R} : \mathrm{CH}^{d,1}X^r \to H^1\big(F, T_r(d+1)\big)$$

which satisfies a certain compatibility with div_v to be recalled in the proof of Theorem 4.2.

Theorem 4.2. *Assume that V is standard over K. Let r be a positive multiple of κ such that $mr = 2d$ is even. Suppose that Conjecture 4.1 holds and that Γ_r has finite exponent. Then $\mathcal{S}\big(F, V^{\otimes r}(-d)\big) = 0$.*

Proof. Let e be the exponent of Γ_r. Fix $v \in S$ not dividing ℓ and a cycle $Z \in \mathrm{CH}^d X_v^r$. By the definition of Γ_r there is a finite set $\{(\mathfrak{z}_i, f_i)\}$ of codimension d cycles $\mathfrak{z}_i$ on $\mathfrak{X}^r$ and rational functions f_i on $\mathfrak{z}_i$ such that $\sum \mathrm{div}_{\mathfrak{z}_i} f_i = eZ$. Since this divisor has no support on the generic fiber X^r, the element $\zeta_{v,Z} := \sum(Z_i, f_i)$ lies in $\mathrm{CH}^{d,1}X^r$ (with Z_i the generic fiber of $\mathfrak{z}_i$) and satisfies

$$\mathrm{div}_w(\zeta_{v,Z}) = \begin{cases} 0 & w \neq v; \\ eZ & w = v. \end{cases} \tag{4.1}$$

By [10, Proposition 3.1] there is an integer f, independent of v and Z, such that $\mathcal{R}(\zeta_{v,Z})$ lies in $\mathcal{S}_{\{v\}}\big(F, T_r(d+1)\big)$ and has image $ef \cdot c_v(Z)$ under the map

$$\mathcal{S}_{\{v\}}\big(F, T_r(d+1)\big) \to H^1_s\big(F_v, T_r(d+1)\big) \cong \mathfrak{T}_v T_r.$$

(The last isomorphism comes from the fact that $\mathcal{I}_v^{\text{pro-}\ell} \cong \mathbf{Z}_\ell(1)$.) Letting v and Z vary, it thus follows from Conjecture 4.1 that the classes $\mathcal{R}(\zeta_{v,Z})$ form a geometric Euler system for $T_r(d+1)$.

By Lemma 3.1 we can write $V^{\otimes r}(-d) = \oplus V_i$ with each V_i irreducible and locally isotropic. The geometric Euler system for $T_r(d+1)$ induces one for the Cartier dual of each V_i, so that by Proposition 3.3 we have $\mathcal{S}(F, V_i) = 0$ for each i. The theorem follows. □

Remark 4.3. It is overkill to require the subgroup Γ_r to contain the local cycles at all $v \in S$; it would suffice to take all places with Frobenius on V sufficiently congruent to an element of GL_V with distinct eigenvalues and with a minimum number of trivial weights on $V^{\otimes r}(-d)$. See [10] for details.

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