KIDA'S FORMULA AND CONGRUENCES

ROBERT POLLACK AND TOM WESTON

1. Introduction

Let f be a modular eigenform of weight at least two and let F be a finite abelian extension of $\mathbf{Q}$. Fix an odd prime p at which f is ordinary in the sense that the p^{th} Fourier coefficient of f is not divisible by p. In Iwasawa theory, one associates two objects to f over the cyclotomic $\mathbf{Z}_p$-extension F_∞ of F: a Selmer group $\operatorname{Sel}(F_\infty, A_f)$ (where A_f denotes the divisible version of the two-dimensional Galois representation attached to f) and a p-adic L-function $L_p(F_\infty, f)$. In this paper we prove a formula, generalizing work of Kida and Hachimori–Matsuno, relating the Iwasawa invariants of these objects over F with their Iwasawa invariants over p-extensions of F.

For Selmer groups our results are significantly more general. Let T be a lattice in a nearly ordinary p-adic Galois representation V; set $A = V/T$. When $\operatorname{Sel}(F_\infty, A)$ is a cotorsion Iwasawa module, its Iwasawa μ-invariant $\mu^{\text{alg}}(F_\infty, A)$ is said to vanish if $\operatorname{Sel}(F_\infty, A)$ is cofinitely generated and its λ-invariant $\lambda^{\text{alg}}(F_\infty, A)$ is simply its p-adic corank. We prove the following result relating these invariants in a p-extension.

Theorem 1. *Let F'/F be a finite Galois p-extension that is unramified at all places dividing p. Assume that T satisfies the technical assumptions (1)–(5) of Section 2. If $\operatorname{Sel}(F_\infty, A)$ is Λ-cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then $\operatorname{Sel}(F'_\infty, A)$ is Λ-cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$. Moreover, in this case*

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w'} m(F'_{\infty,w'}/F_{\infty,w}, V)$$

where the sum extends over places w' of F'_∞ which are ramified in F'_∞/F_∞.

If V is associated to a cuspform f and F' is an abelian extension of $\mathbf{Q}$, then the same results hold for the analytic Iwasawa invariants of f.

Here $m(F'_{\infty,w'}/F_{\infty,w}, V)$ is a certain difference of local multiplicities defined in Section 2.1. In the case of Galois representations associated to Hilbert modular forms, these local factors can be made quite explicit; see Section 4.1 for details.

It follows from Theorem 1 and work of Kato that if the p-adic main conjecture holds for a modular form f over $\mathbf{Q}$, then it holds for f over all abelian p-extensions of $\mathbf{Q}$; see Section 4.2 for details.

These Riemann-Hurwitz type formulas were first discovered by Kida [5] in the context of λ-invariants of CM fields. More precisely, when F'/F is a p-extension of CM fields and $\mu^-(F_\infty/F) = 0$, Kida gave a precise formula for $\lambda^-(F'_\infty/F')$ in terms of $\lambda^-(F_\infty/F)$ and local data involving the primes that ramify in F'/F. (See also [4] for a representation theoretic interpretation of Kida's result.) A similar formula in a somewhat different setting was given for elliptic curves with complex multiplication at ordinary primes by Wingberg [12]; Hachimori–Matsuno [3] established

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the cyclotomic version in general. The analytic analogue was first established for ideal class groups by Sinnott [10] and for elliptic curves by Matsuno [7].

Our proof is most closely related to the arguments in [10] and [7] where congruences implicitly played a large role in their study of analytic λ-invariants. In this paper, we make the role of congruences more explicit and apply these methods to study both algebraic and analytic λ-invariants.

As is usual, we first reduce to the case where F'/F is abelian. (Some care is required to show that our local factors are well behaved in towers of fields; this is discussed in Section 2.1.) In this case, the λ-invariant of V over F' can be expressed as the sum of the λ-invariants of twists of V by characters of $\mathrm{Gal}(F'/F)$. The key observation (already visible in both [10] and [7]) is that since $\mathrm{Gal}(F'/F)$ is a p-group, all of its characters are trivial modulo a prime over p and, thus, the twisted Galois representations are all congruent to V modulo a prime over p. The algebraic case of Theorem 1 then follows from the results of [11] which gives a precise local formula for the difference between λ-invariants of congruent Galois representations. The analytic case is handled similarly using the results of [1].

The basic principle behind this argument is that a formula relating the Iwasawa invariants of congruent Galois representations should imply of a transition formula for these invariants in p-extensions. As an example of this, in Section 4.3, we use results of [2] to prove a Kida formula for the Iwasawa invariants (in the sense of [8, 6, 9]) of weight 2 modular forms at supersingular primes.

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2. Algebraic invariants

2.1. **Local preliminaries.** We begin by studying the local terms that appear in our results. Fix distinct primes ℓ and p and let L denote a finite extension of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_\ell$. Fix a field K of characteristic zero and a finite-dimensional K-vector space V endowed with a continuous K-linear action of the absolute Galois group G_L of L. Set

$$m_L(V) := \dim_K \left(V_{I_L}\right)^{G_L},$$

the multiplicity of the trivial representation in the I_L-coinvariants of V. Note that this multiplicity is invariant under extension of scalars, so that we can enlarge K as necessary.

Let L' be a finite Galois p-extension of L. Note that L' must be cyclic and totally ramified since L contains the $\mathbf{Z}_p$-extension of $\mathbf{Q}_\ell$. Let G denote the Galois group of L'/L. Assuming that K contains all $[L' : L]^{\mathrm{th}}$ roots of unity, for a character $\chi : G \to K^\times$ of G, we set $V_\chi = V \otimes_K K(\chi)$ with $K(\chi)$ a one-dimensional K-vector space on which G acts via χ. We define

$$m(L'/L, V) := \sum_{\chi \in G^\vee} m_L(V) - m_L(V_\chi)$$

where $G^\vee$ denotes the K-dual of G.

The next result shows how these invariants behave in towers of fields.

Lemma 2.1. *Let L'' be a finite Galois p-extension of L and let L' be a Galois extension of L contained in L''. Assume that K contains all $[L'' : L]^{th}$ roots of* *unity. Then*

$$m(L''/L, V) = [L'' : L'] \cdot m(L'/L, V) + m(L''/L', V).$$

Proof. Set $G = \mathrm{Gal}(L''/L)$ and $H = \mathrm{Gal}(L''/L')$. Consider the Galois group $G_L/I_{L''}$ over L of the maximal unramified extension of L''. It sits in an exact sequence

$$0 \to G_{L''}/I_{L''} \to G_L/I_{L''} \to G \to 0 \tag{1}$$

which is in fact split since the maximal unramified extensions of both L and L'' are obtained by adjoining all prime-to-p roots of unity.

Fix a character $\chi \in G^\vee$. We compute

$$\begin{aligned}
m_L(V_\chi) &= \dim_K \big((V_\chi)_{I_L}\big)^{G_L} \\
&= \dim_K \Big(\big(((V_\chi)_{I_{L''}})_G\big)^{G_{L''}}\Big)^G \\
&= \dim_K \Big(\big(((V_\chi)_{I_{L''}})^{G_{L''}}\big)_G\Big)^G \quad \text{since (1) is split} \\
&= \dim_K \Big(\big((V_\chi)_{I_{L''}}\big)^{G_{L''}}\Big)^G \quad \text{since } G \text{ is finite cyclic} \\
&= \dim_K \big((V_{I_{L''}})^{G_{L''}} \otimes \chi\big)^G \quad \text{since } \chi \text{ is trivial on } G_{L''}.
\end{aligned}$$

The lemma thus follows from the following purely group-theoretical statement applied with $W = (V_{I_{L''}})^{G_{L''}}$: for a finite dimensional representation W of a finite abelian group G over a field of characteristic zero containing $\mu_{\#G}$, we have

$$\sum_{\chi \in G^\vee} \big(\langle W, 1\rangle_G - \langle W, \chi\rangle_G\big) = \\ \#H \cdot \sum_{\chi \in (G/H)^\vee} \big(\langle W, 1\rangle_G - \langle W, \chi\rangle_G\big) + \sum_{\chi \in H^\vee} \big(\langle W, 1\rangle_H - \langle W, \chi\rangle_H\big)$$

for any subgroup H of G; here $\langle W, \chi\rangle_G$ (resp. $\langle W, \chi\rangle_H$) is the multiplicity of the character χ in W regarded as a representation of G (resp. H). To prove this, we compute

$$\begin{aligned}
&\sum_{\chi \in G^\vee} \big(\langle W, 1\rangle_G - \langle W, \chi\rangle_G\big) \\
&\quad = \#G \cdot \langle W, 1\rangle_G - \Big\langle W, \mathrm{Ind}_1^G 1\Big\rangle_G \\
&\quad = \#G \cdot \langle W, 1\rangle_G - \#H \cdot \Big\langle W, \mathrm{Ind}_H^G 1\Big\rangle_G + \#H \cdot \Big\langle W, \mathrm{Ind}_H^G 1\Big\rangle_G - \Big\langle W, \mathrm{Ind}_1^G 1\Big\rangle_G \\
&\quad = \#H \cdot \sum_{\chi \in (G/H)^\vee} \big(\langle W, 1\rangle_G - \langle W, \chi\rangle_G\big) + \sum_{\chi \in H^\vee} \Big(\Big\langle W, \mathrm{Ind}_H^G 1\Big\rangle_G - \Big\langle W, \mathrm{Ind}_H^G \chi\Big\rangle_G\Big) \\
&\quad = \#H \cdot \sum_{\chi \in (G/H)^\vee} \big(\langle W, 1\rangle_G - \langle W, \chi\rangle_G\big) + \sum_{\chi \in H^\vee} \big(\langle W, 1\rangle_H - \langle W, \chi\rangle_H\big)
\end{aligned}$$

by Frobenius reciprocity. □

2.2. **Global preliminaries.** Fix a number field F; for simplicity we assume that F is either totally real or totally imaginary. Fix also an odd prime p and a finite extension K of $\mathbf{Q}_p$; we write $\mathcal{O}$ for the ring of integers of K, π for a fixed choice of uniformizer of $\mathcal{O}$, and $k = \mathcal{O}/\pi$ for the residue field of $\mathcal{O}$.

Let T be a nearly ordinary Galois representation over F with coefficients in $\mathcal{O}$; that is, T is a free $\mathcal{O}$-module of some rank n endowed with an $\mathcal{O}$-linear action of the absolute Galois group G_F, together with a choice for each place v of F dividing p of a complete flag

$$0 = T_v^0 \subset T_v^1 \subset \cdots \subset T_v^n = T$$

stable under the action of the decomposition group $G_v \subseteq G_F$ of v. We make the following assumptions on T:

(1) For each place v dividing p we have

$$\left(T_v^i/T_v^{i-1}\right) \otimes k \not\cong \left(T_v^j/T_v^{j-1}\right) \otimes k$$

as $k[G_v]$-modules for all $i \neq j$;

(2) If F is totally real, then $\operatorname{rank} T^{c_v=1}$ is independent of the archimedean place v (here c_v is a complex conjugation at v);

(3) If F is totally imaginary, then n is even.

Remark 2.2. The conditions above are significantly more restrictive than are actually required to apply the results of [11]. As our main interest is in abelian (and thus necessarily Galois) extensions of $\mathbf{Q}$, we have chosen to include the assumptions (2) and (3) to simplify the exposition. The assumption (1) is also stronger then necessary: all that is actually needed is that the centralizer of $T \otimes k$ consists entirely of scalars and that $\mathfrak{gl}_n/\mathfrak{b}_v$ has trivial adjoint G_v-invariants for all places v dividing p; here $\mathfrak{gl}_n$ denotes the p-adic Lie algebra of GL_n and $\mathfrak{b}_v$ denotes the p-adic Lie algebra of the Borel subgroup associated to the complete flag at v. In particular, when T has rank 2, we may still allow the case that $T \otimes k$ has the form

$$\begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}$$

so long as $*$ is non-trivial. (Equivalently, if T is associated to a modular form f, the required assumption is that f is *p-distinguished*.)

Set $A = T \otimes_{\mathcal{O}} K/\mathcal{O}$; it is a cofree $\mathcal{O}$-module of corank n with an $\mathcal{O}$-linear action of G_F. Let c equal the rank of $T_v^{c_v=1}$ (resp. $n/2$) if F is totally real (resp. totally imaginary) and set

$$A_v^{\mathrm{cr}} := \operatorname{im}\bigl(T_v^c \otimes_{\mathcal{O}} K \hookrightarrow T \otimes_{\mathcal{O}} K \twoheadrightarrow A\bigr).$$

We define the Selmer group of A over the cyclotomic $\mathbf{Z}_p$-extension F_∞ of F by

$$\operatorname{Sel}(F_\infty, A) = \ker\left(H^1(F_\infty, A) \to \left(\bigoplus_{w \nmid p} H^1(F_{\infty,w}, A)\right) \times \left(\bigoplus_{w \mid p} H^1(F_{\infty,w}, A/A_v^{\mathrm{cr}})\right)\right).$$

The Selmer group $\operatorname{Sel}(F_\infty, A)$ is naturally a module for the Iwasawa algebra $\Lambda_{\mathcal{O}} := \mathcal{O}[[\operatorname{Gal}(F_\infty/F)]]$. If $\operatorname{Sel}(F_\infty, A)$ is $\Lambda_{\mathcal{O}}$-cotorsion (that is, if the dual of $\operatorname{Sel}(F_\infty, A)$ is a torsion $\Lambda_{\mathcal{O}}$-module), then we write $\mu^{\mathrm{alg}}(F_\infty, A)$ and $\lambda^{\mathrm{alg}}(F_\infty, A)$ for its Iwasawa invariants; in particular, $\mu^{\mathrm{alg}}(F_\infty, A) = 0$ if and only if $\operatorname{Sel}(F_\infty, A)$ is a cofinitely generated $\mathcal{O}$-module, while $\lambda^{\mathrm{alg}}(F_\infty, A)$ is the $\mathcal{O}$-corank of $\operatorname{Sel}(F_\infty, A)$.

Remark 2.3. In the case that T is in fact an *ordinary* Galois representation (meaning that the action of inertia on each T_v^i/T_v^{i-1} is by an integer power e_i (independent of v) of the cyclotomic character such that $e_1 > e_2 > \ldots > e_n$), then our Selmer group $\mathrm{Sel}(F_\infty, A)$ is simply the Selmer group in the sense of Greenberg of a twist of A; see [11, Section 1.3] for details.

2.3. **Extensions.** Let F' be a finite Galois extension of F with degree equal to a power of p. We write F'_∞ for the cyclotomic $\mathbf{Z}_p$-extension of F' and set $G = \mathrm{Gal}(F'_\infty/F_\infty)$. Note that T satisfies hypotheses (1)–(3) over F' as well, so that we may define $\mathrm{Sel}(F'_\infty, A)$ analogously to $\mathrm{Sel}(F_\infty, A)$. (For (1) this follows from the fact that G_v acts on $(T_v^i/T_v^{i-1}) \otimes k$ by a character of prime-to-p order; for (2) and (3) it follows from the fact that p is assumed to be odd.)

Lemma 2.4. *The restriction map*

$$\mathrm{Sel}(F_\infty, A) \to \mathrm{Sel}(F'_\infty, A)^G \tag{2}$$

has finite kernel and cokernel.

Proof. This is straightforward from the definitions and the fact that G is finite and A is cofinitely generated; see [3, Lemma 3.3] for details. □

We can use Lemma 2.4 to relate the μ-invariants of A over F_∞ and F'_∞.

Corollary 2.5. *If* $\mathrm{Sel}(F_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F_\infty, A) = 0$*, then* $\mathrm{Sel}(F'_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F'_\infty, A) = 0$.

Proof. This is a straightforward argument using Lemma 2.4 and Nakayama's lemma for compact local rings; see [3, Corollary 3.4] for details. □

Fix a finite extension K' of K containing all $[F' : F]^{\mathrm{th}}$ roots of unity. Consider a character $\chi : G \to \mathcal{O}'^{\times}$ taking values in the ring of integers $\mathcal{O}'$ of K'; note that χ is necessarily even since $[F' : F]$ is odd. We set

$$A_\chi = A \otimes_{\mathcal{O}} \mathcal{O}'(\chi)$$

where $\mathcal{O}'(\chi)$ is a free $\mathcal{O}'$-module of rank one with G_{F_∞}-action given by χ. If we give A_χ the induced complete flags at places dividing p, then A_χ satisfies hypotheses (1)–(3) and we have

$$A_{\chi,v}^{\mathrm{cr}} = A_v^{\mathrm{cr}} \otimes_{\mathcal{O}} \mathcal{O}'(\chi) \subseteq A_\chi$$

for each place v dividing p. We write $\mathrm{Sel}(F_\infty, A_\chi)$ for the corresponding Selmer group, regarded as a $\Lambda_{\mathcal{O}'}$-module; in particular, by $\lambda^{\mathrm{alg}}(F_\infty, A_\chi)$ we mean the $\mathcal{O}'$-corank of $\mathrm{Sel}(F_\infty, A_\chi)$, rather than the $\mathcal{O}$-corank. We write $G^\vee$ for the set of all characters $\chi : G \to \mathcal{O}'^{\times}$.

Proposition 2.6. *Assume that* $\mathrm{Sel}(F_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F_\infty, A) = 0$. *If G is an abelian group, then there is a natural map*

$$\bigoplus_{\chi \in G^\vee} \mathrm{Sel}(F_\infty, A_\chi) \to \mathrm{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'$$

with finite kernel and cokernel.

Proof. First note that as $\mathcal{O}'[[G_{F'}]]$-modules we have

$$A \otimes_{\mathcal{O}} \mathcal{O}' \cong A_\chi$$

from which it easily follows that

$$\big(\mathrm{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'(\chi)\big)^G = \mathrm{Sel}(F'_\infty, A_\chi)^G. \tag{3}$$

Also, for any cofinitely generated $\mathcal{O}[G]$-module S, the natural map

$$\bigoplus_{\chi \in G^\vee} \big(S \otimes \mathcal{O}'(\chi)\big)^G \to S \otimes \mathcal{O}' \tag{4}$$

has finite kernel and cokernel. Since we are assuming that $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, we may take $S = \mathrm{Sel}(F'_\infty, A)$ in (4); combining this with (3) yields a map

$$\bigoplus_{\chi \in G^\vee} \big(\mathrm{Sel}(F'_\infty, A_\chi)\big)^G \to \mathrm{Sel}(F'_\infty, A_\chi) \otimes \mathcal{O}'$$

with finite kernel and cokernel. Now applying Lemma 2.4 for each twist A_χ, we obtain our proposition. □

As an immediate corollary, we have the following.

Corollary 2.7. *If* $\mathrm{Sel}(F_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F_\infty, A) = 0$*, then each group* $\mathrm{Sel}(F_\infty, A_\chi)$ *is* $\Lambda_{\mathcal{O}'}$*-cotorsion with* $\mu^{\mathrm{alg}}(F_\infty, A_\chi) = 0$*. Moreover, if* G *is abelian, then*

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\mathrm{alg}}(F_\infty, A_\chi).$$

2.4. **Algebraic transition formula.** We continue with the notation of the previous section. We write $R(F'_\infty/F_\infty)$ for the set of prime-to-p places of F'_∞ which are ramified in F'_∞/F_∞. For a place $w' \in R(F'_\infty/F_\infty)$, we write w for its restriction to F_∞.

Theorem 2.8. *Let* F'/F *be a finite Galois* p*-extension with Galois group* G *which is tamely ramified at all places dividing* p*. Let* T *be a nearly ordinary Galois representation over* F *with coefficients in* $\mathcal{O}$ *satisfying (1)–(3). Set* $A = T \otimes K/\mathcal{O}$ *and assume that:*

(4) $H^0(F, A[\pi]) = H^0\big(F, \mathrm{Hom}(A[\pi], \mu_p)\big) = 0$;
(5) $H^0(I_v, A/A_v^{cr})$ *is* $\mathcal{O}$*-divisible for all* v *dividing* p*.*

If $\mathrm{Sel}(F_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F_\infty, A) = 0$*, then* $\mathrm{Sel}(F'_\infty, A)$ *is* Λ*-cotorsion with* $\mu^{\mathrm{alg}}(F'_\infty, A) = 0$*. Moreover, in this case,*

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V)$$

with $V = T \otimes K$ *and* $m(F'_{\infty,w'}/F_{\infty,w}, V)$ *as in Section 2.1.*

Note that $m(F'_{\infty,w'}/F_{\infty,w}, V)$ in fact depends only on w and not on w'. The hypotheses (4) and (5) are needed to apply the results of [11]; they will not otherwise appear in the proof below. We note that the assumption that F'/F is unramified at p is primarily needed to assure that the condition (5) holds for twists of A as well.

Since p-groups are solvable and the only simple p-group is cyclic, the next lemma shows that it suffices to consider the case of $\mathbf{Z}/p\mathbf{Z}$-extensions.

Lemma 2.9. *Let* F''/F *be a Galois* p*-extension of number fields and let* F' *be an intermediate extension which is Galois over* F*. Let* T *be as above. If Theorem 2.8 holds for* T *with respect to any two of the three field extensions* F''/F'*,* F'/F *and* F''/F*, then it holds for* T *with respect to the third extension.*

Proof. This is clear from Corollary 2.5 except for the λ-invariant formula. Substituting the formula for $\lambda(F'_\infty, A)$ in terms of $\lambda(F_\infty, A)$ into the formula for $\lambda(F''_\infty, A)$ in terms of $\lambda(F'_\infty, A)$, one finds that it suffices to show that

$$\sum_{w'' \in R(F''_\infty/F_\infty)} m(F''_{\infty,w''}/F_{\infty,w}, V) = [F''_\infty : F'_\infty] \cdot \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V) + \sum_{w'' \in R(F''_\infty/F'_\infty)} m(F''_{\infty,w''}/F'_{\infty,w'}, V).$$

This formula follows upon summing the formula of Lemma 2.1 over all $w'' \in R(F''_\infty/F_\infty)$ and using the two facts:

- $[F''_\infty : F'_\infty]/[F''_{\infty,w''} : F'_{\infty,w'}]$ equals the number of places of F''_∞ lying over w' (since the residue field of $F_{\infty,w}$ has no p-extensions);
- $m(F''_{\infty,w''}/F'_{\infty,w'}, V) = 0$ for any $w'' \in R(F''_\infty/F_\infty) - R(F''_\infty/F'_\infty)$.

□

Proof of Theorem 2.8. By Lemma 2.9 and the preceding remark, we may assume that F'_∞/F_∞ is a cyclic extension of degree p. The fact that $\mathrm{Sel}(F'_\infty, A)$ is cotorsion with trivial μ-invariant is simply Corollary 2.5. Furthermore, by Corollary 2.7, we have

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\mathrm{alg}}(F_\infty, A_\chi).$$

For $\chi \in G^\vee$, note that χ is trivial modulo a uniformizer π' of $\mathcal{O}'$ as it takes values in μ_p. In particular, the residual representations $A_\chi[\pi']$ and $A[\pi]$ are isomorphic. Under the hypotheses (1)–(5), the result [11, Theorem 1] gives a precise formula for the relation between λ-invariants of congruent Galois representations. In the present case it takes the form:

$$\lambda^{\mathrm{alg}}(F_\infty, A_\chi) = \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \nmid p} \big(m_{F_{\infty,w}}(V \otimes \omega^{-1}) - m_{F_{\infty,w}}(V_\chi \otimes \omega^{-1})\big)$$

where the sum is over all prime-to-p places w' of F'_∞, w denotes the place of F_∞ lying under w' and ω is the mod p cyclotomic character. The only non-zero terms in this sum are those for which w' is ramified in F'_∞/F_∞. For any such w', we have $\mu_p \subseteq F_{\infty,w}$ by local class field theory so that ω is in fact trivial at w; thus

$$\lambda^{\mathrm{alg}}(F_\infty, A_\chi) = \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} \big(m_{F_{\infty,w}}(V) - m_{F_{\infty,w}}(V_\chi)\big).$$

Summing over all $\chi \in G^\vee$ then yields

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V)$$

which completes the proof. □

3. Analytic invariants

3.1. **Definitions.** Let $f = \sum a_n q^n$ be a modular eigenform of weight $k \geq 2$, level N and character ε. Let K denote the finite extension of $\mathbf{Q}_p$ generated by the Fourier coefficients of f (under some fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$), let $\mathcal{O}$ denote the ring of integers of K and let k denote the residue field of $\mathcal{O}$. Let V_f denote a two-dimensional K-vector space with Galois action associated to f in the usual way; thus the characteristic polynomial of a Frobenius element at a prime $\ell \nmid Np$ is

$$x^2 - a_\ell x + \ell^{k-1}\varepsilon(\ell).$$

Fix a Galois stable $\mathcal{O}$-lattice T_f in V_f. We assume that $T_f \otimes k$ is an irreducible Galois representation; in this case T_f is uniquely determined up to scaling. Set $A_f = T_f \otimes K/\mathcal{O}$.

Assuming that f is p-ordinary (in the sense that a_p is relatively prime to p) and fixing a canonical period for f, one can associate to f a p-adic L-function $L_p(\mathbf{Q}_\infty/\mathbf{Q}, f)$ which lies in $\Lambda_{\mathcal{O}}$. This is well-defined up to a p-adic unit (depending upon the choice of a canonical period) and thus has well-defined Iwasawa invariants.

Let $F/\mathbf{Q}$ be a finite abelian extension and let F_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of F. For a character χ of $\mathrm{Gal}(F/\mathbf{Q})$, we denote by f_χ the modular eigenform $\sum a_n \chi(n) q^n$ obtained from f by twisting by χ (viewed as a Dirichlet character). If f is p-ordinary and $F/\mathbf{Q}$ is unramified at p, then f_χ is again p-ordinary and we define

$$L_p(F_\infty/F, f) = \prod_{\chi \in \mathrm{Gal}(F/\mathbf{Q})^\vee} L_p(\mathbf{Q}_\infty/\mathbf{Q}, f_\chi).$$

If $F/\mathbf{Q}$ is ramified at p, it is still possible to define $L_p(F_\infty/F, f)$; see [7, pg. 5], for example.

If F_1 and F_2 are two distinct number fields whose cyclotomic $\mathbf{Z}_p$-extensions agree, the corresponding p-adic L-functions of f over F_1 and F_2 need not agree. However, it is easy to check that the λ-invariants of these two power series are equal while their μ-invariants differ by a factor of a power of p. As we are only interested in the case of vanishing μ-invariants, we will abuse notation somewhat and simply denote the Iwasawa invariants of $L_p(F_\infty/F, f)$ by $\mu^{\mathrm{an}}(F_\infty, f)$ and $\lambda^{\mathrm{an}}(F_\infty, f)$.

3.2. **Analytic transition formula.** Let $F/\mathbf{Q}$ be a finite abelian extension of $\mathbf{Q}$ and let F' be a finite p-extension of F such that $F'/\mathbf{Q}$ is abelian. As always, let F_∞ and F'_∞ denote the cyclotomic $\mathbf{Z}_p$-extensions of F and F'. As before, we write $R(F'_\infty/F_\infty)$ for the set of prime-to-p places of F'_∞ which are ramified in F'_∞/F_∞.

We begin with the following observation.

Lemma 3.1. *Let $F/\mathbf{Q}$ be a finite abelian extension. Then there exists a finite abelian extension $K/\mathbf{Q}$ such that:*

- *$F_\infty = K_\infty$;*
- *Every character of $\mathrm{Gal}(K/\mathbf{Q})$ is a product of a power of the mod p cyclotomic character and a character unramified at p.*

Proof. Write the conductor of F as $p^a m$ with m relatively prime to p; if $a = 0$ then we may take $K = F$, so we assume $a \geq 1$. Let H denote the subgroup of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^a m})/\mathbf{Q})$ corresponding to the unique cyclic subgroup of order p^{a-1} of the first factor in the decomposition

$$\mathrm{Gal}(\mathbf{Q}(\mu_{p^a m})/\mathbf{Q}) \cong (\mathbf{Z}/p^a\mathbf{Z})^\times \times (\mathbf{Z}/m\mathbf{Z})^\times.$$

Let K denote the fixed field of the image of H in $\mathrm{Gal}(F/\mathbf{Q})$. Let $n \geq 1$ be such that $\mathbf{Q}_n = \mathbf{Q}_\infty \cap F$. Note that K satisfies $K\mathbf{Q}_n = F$ and $K \cap \mathbf{Q}_n = \mathbf{Q}$. In particular, $K_\infty = F_\infty$, as desired.

Let K' denote the maximal subfield of K such that $K'/\mathbf{Q}$ is unramified outside p. To complete the proof it suffices to show that $K(\mu_p) = K'(\mu_p)$. Since $K'(\mu_{p^\infty})$ is ramified at p we must have $K(\mu_{p^\infty}) = K'(\mu_{p^\infty})$.

It follows that for some $m \geq 1$ we have $K(\mu_{p^m}) = K'(\mu_{p^m})$. Let m be the least such integer and assume $m > 1$. Since $K'/\mathbf{Q}$ is unramified at p, $K'(\mu_{p^{m-1}})$ is the unique index p subfield of $K'(\mu_{p^m})$ over which $K'(\mu_{p^m})$ is ramified at p. On the other hand, $[K(\mu_{p^m}) : K(\mu_{p^{m-1}})] = p$ since $K \cap \mathbf{Q}_n = \mathbf{Q}$. It follows that we must have $K(\mu_{p^m}) = K'(\mu_{p^m})$, which contradicts the minimality of m and thus implies that $m = 1$, as desired. □

Theorem 3.2. *Let f be a p-ordinary modular form such that $T_f \otimes k$ is irreducible and p-distinguished. If $\mu^{\mathrm{an}}(F_\infty, f) = 0$, then $\mu^{\mathrm{an}}(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda^{\mathrm{an}}(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{an}}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V_f).$$

Proof. By Lemma 2.9, we may assume $[F : \mathbf{Q}]$ is prime-to-p. Indeed, let F_0 be the maximal subfield of F of prime-to-p degree over $\mathbf{Q}$. By Lemma 2.9, knowledge of the theorem for the two extensions F'/F_0 and F/F_0 would then imply it for F'/F as well. Applying Lemma 3.1 we may further assume that every character of $\mathrm{Gal}(F/\mathbf{Q})$ and $\mathrm{Gal}(F'/\mathbf{Q})$ is the product of a power of the mod p cyclotomic character and a character unramified at p.

After making these reductions, we let M denote the (unique) p-extension of $\mathbf{Q}$ inside of F' such that $MF = F'$. Set $G = \mathrm{Gal}(F/\mathbf{Q})$ and $H = \mathrm{Gal}(M/\mathbf{Q})$, so that $\mathrm{Gal}(F'/\mathbf{Q}) \cong G \times H$. We have

$$\mu^{\mathrm{an}}(F_\infty, f) = \sum_{\psi \in \mathrm{Gal}(F/\mathbf{Q})^\vee} \mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) \tag{5}$$

and

$$\mu^{\mathrm{an}}(F'_\infty, f) = \sum_{\psi \in \mathrm{Gal}(F'/\mathbf{Q})^\vee} \mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_{\psi\chi}). \tag{6}$$

Since we are assuming that $\mu^{\mathrm{an}}(F_\infty, f) = 0$ and since these μ-invariants are non-negative, from (5) it follows that $\mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) = 0$ for each $\psi \in \mathrm{Gal}(F/\mathbf{Q})^\vee$.

Fix $\psi \in G^\vee$. For any $\chi \in H^\vee$, $\psi\chi$ is congruent to ψ modulo any prime over p and thus f_χ and $f_{\psi\chi}$ are congruent modulo any prime over p. Then, since $\mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) = 0$, by [1, Theorem 3.7.5] it follows that $\mu^{\mathrm{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) = 0$ for each $\chi \in H^\vee$. (Note that the results of [1] apply to twists of p-ordinary forms by powers of the mod p cyclotomic character; this is why the reduction provided by Lemma 3.1 is necessary for this argument.) Therefore, by (6) we have that $\mu^{\mathrm{an}}(F'_\infty, f) = 0$ proving the first part of the theorem.

For λ-invariants, we again have

$$\lambda^{\mathrm{an}}(F_\infty, f) = \sum_{\psi \in \mathrm{Gal}(F/\mathbf{Q})^\vee} \lambda^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi).$$

and

$$\lambda^{\mathrm{an}}(F'_\infty, f) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \lambda^{\mathrm{an}}(\mathbf{Q}_\infty, f_{\psi\chi}). \tag{7}$$

By [1, Theorem 3.7.7] the congruence between f_χ and $f_{\psi\chi}$ implies that

$$\lambda^{\mathrm{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) - \lambda^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_{v' \in R(M_\infty/\mathbf{Q}_\infty)} \big(m_{\mathbf{Q}_{\infty,v}}(V_{f_{\psi\chi}} \otimes \omega^{-1}) - m_{\mathbf{Q}_{\infty,v}}(V_{f_\psi} \otimes \omega^{-1}) \big)$$

where v denotes the place of $\mathbf{Q}_\infty$ lying under the place v' of M_∞. Note that in [1] the sum extends over all prime-to-p places; however, the terms are trivial unless χ is ramified at v. Also note that the mod p cyclotomic characters that appear are actually trivial since if $\mathbf{Q}_{\infty,v}$ has a ramified Galois p-extensions for $v \nmid p$, then $\mu_p \subseteq \mathbf{Q}_{\infty,v}$.

Combining this with (7) and the definition of $m(M_{\infty,v'}/\mathbf{Q}_{\infty,v}, V_{f_\psi})$, we conclude that

$$\begin{aligned}
\lambda^{\mathrm{an}}(F'_\infty, f) &= \sum_{\psi \in G^\vee} \left([F'_\infty : F_\infty] \cdot \lambda^{\mathrm{an}}(\mathbf{Q}_\infty, f_\psi) + \sum_{v' \in R(M_\infty/\mathbf{Q}_\infty)} m(M_{\infty,v'}/\mathbf{Q}_{\infty,v}, V_{f_\psi}) \right) \\
&= [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{an}}(F_\infty, f) + \sum_{v' \in R(M_\infty/\mathbf{Q}_\infty)} \sum_{\psi \in G^\vee} m(M_{\infty,v'}/\mathbf{Q}_{\infty,v}, V_{f_\psi}) \\
&= [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{an}}(F_\infty, f) + \sum_{v' \in R(M_\infty/\mathbf{Q}_\infty)} g_{v'}(F'_\infty/M_\infty) \cdot \\
&\qquad m(M_{\infty,v'}/\mathbf{Q}_{\infty,v}, \mathbf{Z}[\mathrm{Gal}(F_{\infty,w}/\mathbf{Q}_{\infty,v})] \otimes V_f)
\end{aligned}$$

where $g_{v'}(F'_\infty/M_\infty)$ denotes the number of places of F'_∞ above the place v' of M_∞. By Frobenius reciprocity,

$$m(M_{\infty,v'}/\mathbf{Q}_{\infty,v}, \mathbf{Z}[\mathrm{Gal}(F_{\infty,w}/\mathbf{Q}_{\infty,v})] \otimes V_f) = m(F'_{\infty,w'}/F_{\infty,w}, V_f)$$

where w' is the unique place of F'_∞ above v' and w. It follows that

$$\lambda(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{an}}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V_f)$$

as desired. □

4. Additional Results

4.1. **Hilbert modular forms.** We illustrate our results in the case of the two-dimensional representation V_f associated to a Hilbert modular eigenform f over a totally real field F. Although in principle our analytic results should remain true in this context, we focus on the less conjectural algebraic picture. Fix a G_F-stable lattice $T_f \subseteq V_f$ and let $A_f = T_f \otimes K/\mathcal{O}$.

Let F' be a finite Galois p-extension of F unramified at all places dividing p; for simplicity we assume also that F' is linearly disjoint from F_∞. Let v be a place of F not dividing p and fix a place v' of F' lying over v. For a character φ of G_v, we define

$$h(\varphi) = \begin{cases} -1 & \varphi \text{ ramified, } \varphi|_{G_{v'}} \text{ unramified, and } \varphi \equiv 1 \bmod \pi; \\ 0 & \varphi \not\equiv 1 \bmod \pi \text{ or } \varphi|_{G_{v'}} \text{ ramified;} \\ e_v(F'/F) - 1 & \varphi \text{ unramified and } \varphi \equiv 1 \bmod \pi \end{cases}$$

where $e_v(F'/F)$ denotes the ramification index of v in F'/F and $G_{v'}$ is the decomposition group at v'. Set

$$h_v(f) = \begin{cases} h(\varphi_1) + h(\varphi_2) & f \text{ principal series with characters } \varphi_1, \varphi_2 \text{ at } v; \\ h(\varphi) & f \text{ special with character } \varphi \text{ at } v; \\ 0 & f \text{ supercuspidal or extraordinary at } v. \end{cases}$$

For example, if f is unramified principal series at v with Frobenius characteristic polynomial

$$x^2 - a_v x + c_v,$$

then

$$h_v(f) = \begin{cases} 2(e_v(F'/F) - 1) & a_v \equiv 2, c_v \equiv 1 \bmod \pi \\ e_v(F'/F) - 1 & a_v \equiv c_v + 1 \not\equiv 2 \bmod \pi \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Assume that f is ordinary (in the sense that for each place v dividing p the Galois representation V_f has a unique one-dimensional quotient unramified at v) and that*

$$H^0(F, A_f[\pi]) = H^0\big(F, \mathrm{Hom}(A_f[\pi], \mu_p)\big) = 0.$$

If $\mathrm{Sel}(F_\infty, A_f)$ is Λ-cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A_f) = 0$, then also $\mathrm{Sel}(F'_\infty, A_f)$ is Λ-cotorsion with $\mu^{\mathrm{alg}}(F'_\infty, A_f) = 0$ and

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_v g_v(F'_\infty/F) \cdot h_v(f);$$

here the sum is over the prime-to-p places of F ramified in F'_∞ and $g_v(F'_\infty/F)$ denotes the number of places of F'_∞ lying over such a v.

Proof. Fix a place v of F not dividing p and let w denote a place of F_∞ lying over v. Since there are exactly $g_v(F_\infty/F)$ such places, by Theorem 2.8 it suffices to prove that

$$(8) \quad h_v(f) = m(F'_{\infty,w'}/F_{\infty,w}, V_f) := \sum_{\chi \in \mathrm{Gal}(F'_{\infty,w'}/F_{\infty,w})^\vee} \left(m_{F_{\infty,w}}(V_f) - m_{F_{\infty,w}}(V_{f,\chi}) \right).$$

This is a straightforward case analysis. We will discuss the case that V_f is special associated to a character φ at v; the other cases are similar. In the special case, we have

$$V_{f,\chi}|_{I_{F_{\infty,w}}} = \begin{cases} K'(\chi\varphi) & \chi\varphi|_{G_{F_{\infty,w}}} \text{ unramified;} \\ 0 & \chi\varphi|_{G_{F_{\infty,w}}} \text{ ramified.} \end{cases}$$

Since an unramified character has trivial restriction to $G_{F_{\infty,w}}$ if and only if it has trivial reduction modulo π, it follows that

$$m_{F_{\infty,w}}(V_{f,\chi}) = \begin{cases} 1 & \varphi \equiv 1 \bmod \pi \text{ and } \chi\varphi|_{G_{F_{\infty,w}}} \text{ unramified;} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the sum in (8) is zero if $\varphi \not\equiv 1 \bmod \pi$ or if φ is ramified when restricted to $G_{F'_{\infty,w'}}$ (as then $\chi\varphi$ is ramified for all $\chi \in G_v^\vee$). If $\varphi \equiv 1 \bmod \pi$ and φ itself is unramified, then $m_{F_{\infty,w}}(V_f) = 1$ while $m_{F_{\infty,w}}(V_{f,\chi}) = 0$ for $\chi \neq 1$, so that the sum in (8) is $[F'_{\infty,w'} : F_{\infty,w}] - 1 = e_v(F'/F) - 1$, as desired. Finally, if $\varphi \equiv 1 \bmod \pi$ and φ is ramified but becomes unramified when restricted to $G_{v'}$, then $m_{F_{\infty,w}}(V_f) = 0$, while $m_{F_{\infty,w}}(V_{f,\chi}) = 1$ for a unique χ, so that the sum is -1. □

Suppose finally that f is in fact the Hilbert modular form associated to an elliptic curve E over F. The only principal series which occur are unramified and we have $c_v \equiv 1 \pmod{\pi}$ (since the determinant of V_f is cyclotomic and F_∞ has a p-extension (namely, F'_∞) ramified at v), so that

$$h_v(f) \neq 0 \quad \Leftrightarrow \quad a_v \equiv 2 \quad \Leftrightarrow \quad E(F_v) \text{ has a point of order } p$$

in which case $h_v(f) = 2(e_v(F'/F) - 1)$. The only characters which may occur in a special constituent are trivial or unramified quadratic, and we have $h_v(f) = e_v(F'/F) - 1$ or 0 respectively. Thus Theorem 4.1 recovers [3, Theorem 3.1] in this case.

4.2. **The main conjecture.** Let f be a p-ordinary elliptic modular eigenform of weight at least two and arbitrary level with associated Galois representation V_f. Let F be a finite abelian extension of $\mathbf{Q}$ with cyclotomic $\mathbf{Z}_p$-extension F_∞. Recall that the p-adic Iwasawa main conjecture for f over F asserts that the Selmer group $\mathrm{Sel}(F_\infty, A_f)$ is Λ-cotorsion and that the characteristic ideal of its dual is generated by the p-adic L-function $L_p(F_\infty, f)$. In fact, when the residual representation of V_f is absolutely irreducible, it is known by work of Kato that $\mathrm{Sel}(F_\infty, A_f)$ is indeed Λ-cotorsion and that $L_p(F_\infty, f)$ is an element of the characteristic ideal of $\mathrm{Sel}(F_\infty, A_f)$. In particular, this reduces the verification of the main conjecture for f over F to the equality of the algebraic and analytic Iwasawa invariants of f over F. The identical transition formulae in Theorems 2.8 and 3.2 thus yield the following immediate application to the main conjecture.

Theorem 4.2. *Let F'/F be a finite p-extension with F' abelian over $\mathbf{Q}$. If the residual representation of V_f is absolutely irreducible and p-distinguished, then the main conjecture holds for f over F with $\mu(F_\infty, f) = 0$ if and only if it holds for f over F' with $\mu(F'_\infty, f) = 0$.*

For an example of Theorem 4.2, consider the eigenform

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$$

of weight 12 and level 1. We take $p = 11$. It is well known that Δ is congruent modulo 11 to the newform associated to the elliptic curve $X_0(11)$. The 11-adic main conjecture is known for $X_0(11)$ over $\mathbf{Q}$; it has trivial μ-invariant and λ-invariant equal to 1 (see, for instance, [1, Example 5.3.1]. We should be clear here that the non-triviality of λ in this case corresponds to a trivial zero of the p-adic L-function; we are using the Greenberg Selmer group which does account for the trivial zero.) It follows from [1] that the 11-adic main conjecture also holds for Δ over $\mathbf{Q}$, again with trivial μ-invariant and λ-invariant equal to 1. Theorem 4.2 thus allows us to conclude that the main conjecture holds for Δ over any abelian 11-extension of $\mathbf{Q}$.

For a specific example, consider $F = \mathbf{Q}(\zeta_{23})^+$; it is a cyclic 11-extension of $\mathbf{Q}$. We can easily use Theorem 4.1 to compute its λ-invariant: using that $\tau(23) = 18643272$ one finds that $h_{23}(\Delta) = 0$, so that $\lambda(\mathbf{Q}(\zeta_{23})^+, \Delta) = 11$.

For a more interesting example, take F to be the unique subfield of $\mathbf{Q}(\zeta_{1123})$ which is cyclic of order 11 over $\mathbf{Q}$. In this case we have

$$\tau(1123) \equiv 2 \pmod{11}$$

so that we have $h_{1123}(\Delta) = 20$. Thus, in this case, Theorem 4.1 shows that $\lambda(F, \Delta) = 31$.

4.3. **The supersingular case.** As mentioned in the introduction, the underlying principle of this paper is that the existence of a formula relating the λ-invariants of congruent Galois representations should imply a Kida-type formula for these invariants. We illustrate this now in the case of modular forms of weight two that are supersingular at p.

Let f be an eigenform of weight 2 and level N with Fourier coefficients in K some finite extension of $\mathbf{Q}_p$. Assume further than $p \nmid N$ and that $a_p(f)$ is not a p-adic unit. In [8], Perrin-Riou associates to f a pair of algebraic and analytic μ-invariants over $\mathbf{Q}_\infty$ which we denote by $\mu^\star_\pm(\mathbf{Q}_\infty, f)$. (Here $\star$ denotes either "alg" or "an" for algebraic and analytic respectively.) Moreover, when $\mu^\star_+(\mathbf{Q}_\infty, f) = \mu^\star_-(\mathbf{Q}_\infty, f)$ or when $a_p(f) = 0$, she also defines corresponding λ-invariants $\lambda^\star_\pm(\mathbf{Q}_\infty, f)$. When $a_p(f) = 0$ these invariants coincide with the Iwasawa invariants of [6] and [9]. We also note that in [8] only the case of elliptic curves is treated, but the methods used there generalize to weight two modular forms.

We extend the definition of these invariants to the cyclotomic $\mathbf{Z}_p$-extension of an unramified abelian extension F of $\mathbf{Q}$. We define

$$\mu^\star_\pm(F_\infty, f) = \sum_{\psi \in \mathrm{Gal}(F/\mathbf{Q})^\vee} \mu^\star_\pm(\mathbf{Q}_\infty, f_\psi) \quad \text{and} \quad \lambda^\star_\pm(F_\infty, f) = \sum_{\psi \in \mathrm{Gal}(F/\mathbf{Q})^\vee} \lambda^\star_\pm(\mathbf{Q}_\infty, f_\psi)$$

for $\star \in \{\mathrm{alg}, \mathrm{an}\}$.

The following transition formula follows from the congruence results of [2].

Theorem 4.3. *Let f be as above and consider a p-extension of number fields F'/F with $F'/\mathbf{Q}$ unramified at p. If $\mu^\star_\pm(F_\infty, f) = 0$, then $\mu^\star_\pm(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda^\star_\pm(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^\star_\pm(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty,w'}/F_{\infty,w}, V_f).$$

In particular, if the main conjecture is true for f over F (with $\mu^\star_\pm(F_\infty, f) = 0$), then the main conjecture is true for f over F' (with $\mu^\star_\pm(F'_\infty, f) = 0$).

Proof. The proof of this theorem proceeds along the lines of the proof of Theorem 3.2 replacing the appeals to the results of [1, 11] to the results of [2]. The main result of [2] is a formula relating the $\lambda^\star_\pm$-invariants of congruent supersingular weight two modular forms. This formula has the same shape as the formulas that appear in [1] and [11] which allows for the proof to proceed nearly verbatim. □

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(Robert Pollack) Department of Mathematics, Boston University, Boston, MA

(Tom Weston) Dept. of Mathematics, University of Massachusetts, Amherst, MA
E-mail address, Robert Pollack: `rpollack@math.bu.edu`
E-mail address, Tom Weston: `weston@math.umass.edu`